HIGH SCHOOL MATHEMATICS

FIRST COURSE

TEACHERS' EDITION

UNIT FOUR 1957-58

GRAPHS AND ORDERED PAIRS

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

Unit 4

GRAPHS AND ORDERED PAIRS

4.01 Locating positions.

A name for something can tell what it is or where it is.

Two numbers can tell you where something is in a plane.

If you have a first number and a second number, you have an ordered pair of numbers.

You can use ordered pairs of numbers to figure "chances" when throwing dice.

4.02 Plane lattices.

Ordered pairs of numbers can be coordinates of points.

A plane lattice can have indefinitely many points.

In a plane lattice with a finite number of points you can count the number of points in the graph of '$x - 2y \leq 7$'.

The intersection is the set of points in both sets; the union is the set of points in either set.

You can shorten the description of a set by using pronumerals.

A rule like '$(y, x) \rightarrow (y + 1, x + 1)$' tells how to make a move in a "plane lattice game".

4.03 The complete coordinate plane.

To get a complete coordinate plane, "fill in" a lattice plane.

Between every two points in the coordinate plane, there is a distance; there is also "paper-and-pencil-distance" in a picture of a coordinate plane.

You know a triangle in the coordinate plane if you know three vertices.

We think of the coordinate plane as cut up into four quadrants.

You make a problem sensible by telling the domain of the pronumeral you are using.

UICSM
University High School
Urbana, Illinois
1955

TEACHERS COMMENTARY

Introduction

This unit is an introduction to coordinate geometry, and is important preparation for the geometry work in SECOND COURSE. One of the inadequacies in teaching elementary graphing in conventional courses is the very small number of types of problems which students can be expected to do. First, you teach them to plot points, and here is your first teaching problem. In order to build in the students the reflex of "go <u>over</u> with the first component, go <u>up</u> or <u>down</u> with the second component", you need to have them actually <u>plot</u> quite a few points. Students of moderate or high ability often resent (justifiably) such a routine kind of job, but they come to class still not sure of whether to go up or to go over with the first component. We think we have a remedy for this.

Next, you usually have students draw the loci of first-degree equations. What else can you do with beginning students? They can graph $\{(x, y): y = x^2\}$, and make a few explorations into the loci of other non-linear equations, but soon the algebra is over their heads. To some extent, this is a problem with all coordinate geometry customarily taught in the beginning course; there is just not enough interesting content. We think you will see in this unit quite a contrasting picture. You will find a great variety of exercises, and you will see many interesting roads down which you and the class will want to travel.

Suppose you give students (who understand the notation) the problem of drawing the locus of:

$$|x| + |y| > 3.$$

Unless the students are accustomed to problems like this, it will be a tough one for them. Many students would give up at the outset and tell you that they can't do it. What can you do to overcome this initial I-can't-do-it attitude? You may tell your students to select specific points and try them. Try (0, 0); does it work? Try (1, 2); does it work? Try (7, 19); does it work? After a student has made a number

of trials with some points accepted and some points rejected, he is in a position to look for patterns and generalities in order to find the full locus. The fact that ultimately he must find an infinite set of points discourages the trial and error beginning. What good is trial and error when you are looking for thousands of points? To get away from this difficulty, we begin with finite loci. Perhaps there are only 25 points in the space in which he is working! If absolutely necessary, he could graph by trial and error completely; he could try every point! Therefore, we can give him problems which are really challenging and are within his range. Of course, most students will find the pattern long before they try every point. You will see that this work with finite lattices gives the student real variety in his exercises.

There are many things behind the scenes in this unit and you will find these things discussed in the Teachers Commentary. We shall point out how this work is preparing for SECOND COURSE to help you in assigning emphasis. As in Unit 3, we shall ask you to introduce new notation concerning sets and set-operations which will permit SECOND COURSE to start in high gear. Also, we shall supply you with many supplementary exercises which should develop proficiency in manipulative skills.

Students may ask in such a discussion why building 1 was not the building in the upper left-hand corner. You should tell them that we are heading toward a system which is similar to this array and that we are trying to establish the appropriate habits. Also, tell them that the building in either right-hand corner could be 1 except for the convention that the order should be from left-to-right as in reading.

* * *

An interesting classroom excursion is to ask students to imagine that the diagram was extended ''upward'' indefinitely far, with five buildings to a row. Then ask students to describe the location of these buildings: 26, 34, 47, 96, 105, 2007, 1000000. This provides them with a bit of practice in arithmetic with numbers modulo 5.

The 25-buildings problem is preparation for work with coordinates in a finite rectangular array of points. Also, the diagram is an interest-catcher.

* * *

A brief discussion of the two types of names is of interest. There is a case for calling a building 'the tool-and-die building' because if you know this name, you know something about what goes on in the building, even though you don't learn anything about its location from the name. And, of course, as is mentioned, there is a real case for calling a building '14'. But, when we abstract from buildings to points, the "tool and die" kind of name loses out entirely. Every point is alike and a point has no properties assigned separately to it. We only want to know where it is. Therefore, a name which tells you location is the obvious choice.

* * *

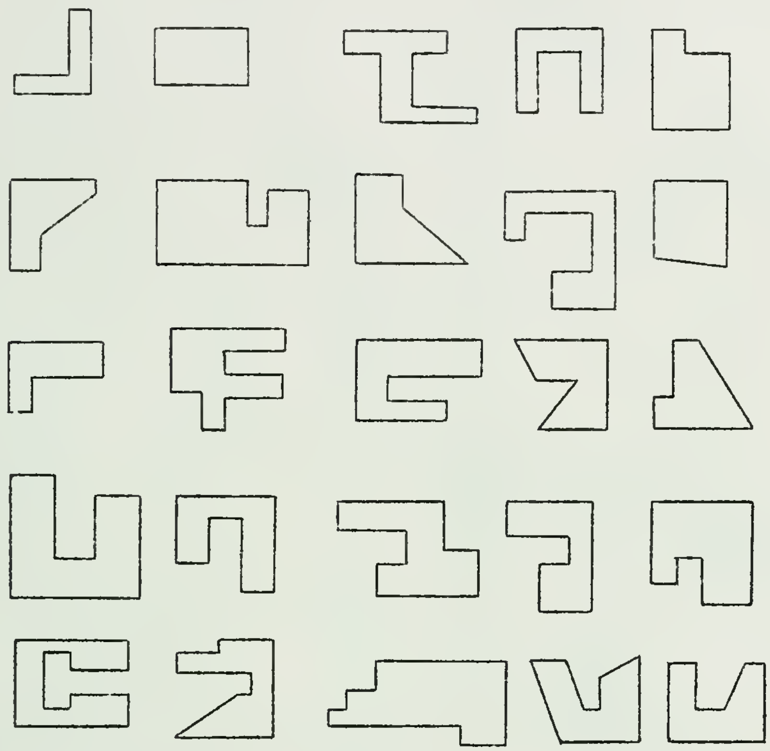
Ask your students what would be the disadvantage in numbering the buildings like this:

21, 5, 14, 9, 1
4, 8, 15, 6, 3
etc.

They will tell you immediately that it would be too hard to remember such a system. Point out to them that they have already learned their number names in a certain order, and that in the arrangement of these 25 buildings, they can also see various kinds of order. They should want these two orders to agree in some fashion.

(continued on T. C. 1B)

4.01 Locating positions.--Suppose that a big manufacturing plant has 25 buildings on its grounds. A map of the grounds looks like:



Your job is to make up 25 names for these buildings so that a newcomer to the plant could find his way around as quickly and easily as possible. Of course, you could just make up 25 different names like "tool and die building" and use those. Such names would tell a newcomer something about what went on in the buildings, but they would not help him learn where the buildings were located. A simpler set of names to remember would be the numerals for whole numbers from 1 through 25. It would be easier to use these names if you assigned them in order, say, like this:

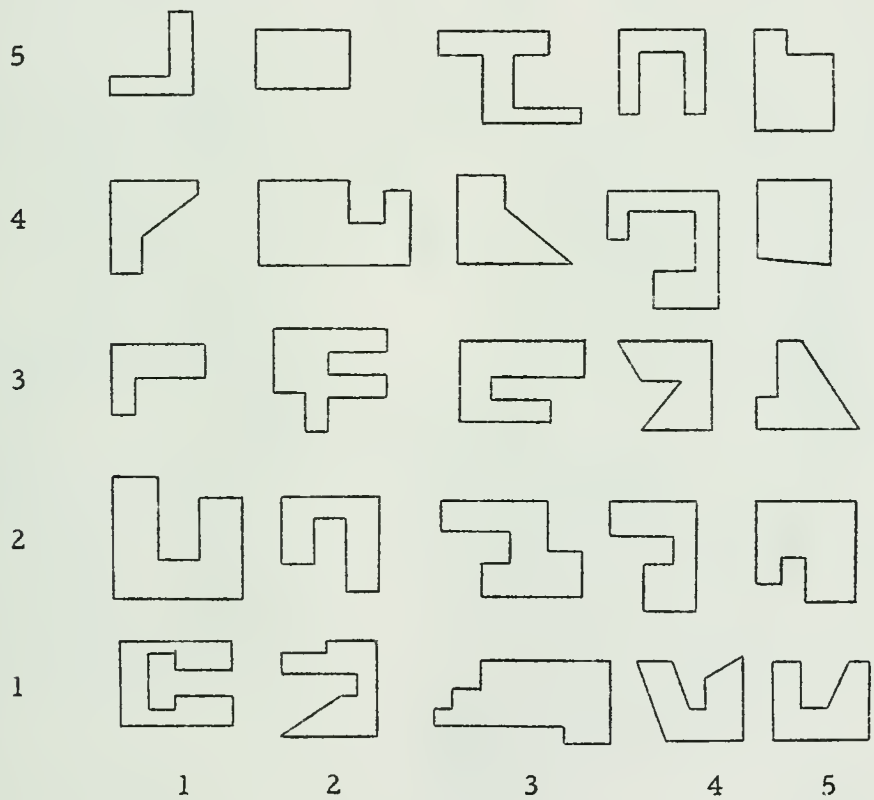
21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

A key sentence is the second one on the page. You know that in one dimension you use single numbers as coordinates, in two dimensions you use ordered pairs of numbers as coordinates, and in three dimensions you use ordered triples of numbers as coordinates. It should seem entirely natural to the student that when given 14, he breaks it down into 4 and 3 when he wants to find the building. So, we use pairs of numbers for points to avoid the necessity of changing from one number to two numbers each time we want to describe location. From this comes the idea that to every point in the plane, there corresponds an ordered pair of numbers, and conversely.

* * *

Again, our particular choice of order here is suggestive of the first quadrant of a coordinate plane. Tell the student who wonders about this choice that he will see it fit into a larger scheme later in the unit. Of course, the larger scheme involves an arbitrary choice, but the student must follow the convention if he is to be able to tell what other people are talking about. Stress the notion of convention in determining order.

Then, it would be very easy to learn where the buildings were located. If you told a newcomer to go to building 14, he would think: "Building 14; there are 5 buildings in each row so the first row is 1 to 5, the second row is 6 to 10, and the third row is 11 to 15. So building 14 is the fourth building in the third row." Notice that to tell himself the location of building 14 he went from the single number 14 to two numbers: the fourth (4) building in the third (3) row. Since you think of the buildings in terms of "which row" and "which building in the row", you might just as well have named them that way in the first place, like this:



Then, by giving someone two numbers, that is, a pair of numbers, you can tell him precisely which building you are talking about. But, note that if you just tell him, say, "4 and 1", he will not know whether you mean, "fourth row, first building", or "fourth building, first row". You could avoid this difficulty by always telling which part of your instruction gives the row, and which part gives the building in that row.

* * *

In Part B, the student is getting preparation for the idea of the Cartesian product of two sets. The Cartesian product of a set and a set (they need not be sets of numbers) is the set which consists of all possible ordered pairs containing first components belonging to the first set and second components belonging to the second set. The students should actually write out the names for all 12 pairs.

Concerning the second sentence, "You could decide ...". The student would probably decide in the reverse fashion if left a free choice. He would pick the row and then the building within the row. Our choice again is consistent with the convention for the number plane, and also shows the students that you can get along perfectly well with a seemingly less natural choice of order.

* * *

Get the students to use the phrase 'ordered pair of numbers'. This is an all-pervading concept in mathematics. The students will see 'ordered pair' come up over and over again and will see them used for a variety of purposes.

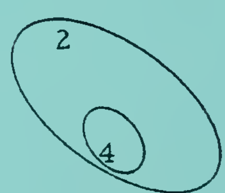
You might want to have students propose other kinds of names for ordered pairs. For example, for (4, 2), they might use:

$$\boxed{\begin{array}{c|c} 4 & 2 \end{array}} \;,$$

or: $$4 \rightarrow 2 \;,$$

or: $$\boxed{\begin{array}{c} 4 \\ \hline 2 \end{array}} \;,$$

or:



After this much discussion, the student may have gained enough objectivity to realize that when he writes a fraction.

$$\frac{2}{5} \,,$$

he is really writing the name of an ordered pair of numbers.

(continued on T. C. 3B)

You could decide that the first number given would always give the building in the row and the second number given would always give the row. If everyone agreed to follow this convention, then it would be enough to say, for example, "Go to 3, 5" or, for another building, "Go to 5, 3."

When you give directions in this manner, you are giving a pair of numbers in order, a first number and a second number. Such a pair of numbers is called an ordered pair of numbers. In mathematics, one customary way of naming an ordered pair of numbers is to put names for the numbers between parentheses, with the name of the first number on the left and the name of the second number on the right and with a comma between the two names to separate them. Following this system we see that the name for the ordered pair of numbers where 4 is the first number and 2 is the second number is:

$$(4, 2).$$

There are many other ways that ordered pairs may be named, but this is the way that will be used in this unit.

EXERCISES

A. Make 16 dots on a sheet of paper in a four-by-four square array. Label the columns '1', '2', '3', '4' from left to right; label the rows in the same way from bottom to top. Agree that in an ordered pair of numbers the first number will tell you the column and the second number will tell you the row. Practice using names of ordered pairs of numbers by finding the point which corresponds to:

1. (3, 2)	2. (2, 3)	3. (4, 1)
4. (3, 3)	5. (1, 2)	6. (2, 1)
7. (8 - 6, 3)	8. (4, 10 - 6)	9. $(2 \div 2, 16 \div (8 \times 2))$

B. Write names for all the ordered pairs of numbers that there are with first numbers chosen from the set of numbers:

2, 3, 7

and with second numbers chosen from the set of numbers:

1, 2, 3, 4.

involved and can proceed with the next step. We say in short: "When you throw a die, you obtain a number."

* * *

An important continuation of the use of dice occurs on page 4-6. We have been asked, usually in jest, if dice-throwing constitutes appropriate subject matter for high school students. If you should be challenged on this question, you might point out that dice are used in a wide variety of quite innocent games played by children [Parchesi, Monopoly, etc.]. Moreover, we are using dice as convenient equipment for a little bit of experimental "research" in probability theory; what other people may do with dice is their problem! However, in obtaining dice to illustrate Part D or to carry out the instructions in Part G on page 4-6, the school or the teacher should purchase them from a local toy or sporting goods store rather than expect students to provide them. The dice should be kept in class as "laboratory" equipment--no homework, please!

In Part C the student should generalize to the idea that the number of ordered pairs he can form is the product of the number of elements in one set by the number of elements in the other set.

The knowledge of "how many ordered pairs" that he gains in Part C will help the student with various excursions into probability that he will make from time to time.

* * *

Part D begins a development in which we are very interested. It has several purposes. First, it illustrates in a striking manner the distinction between an ordered pair of numbers and an unordered pair. If you say the names of two numbers, you must say them in order with respect to time. Therefore, there is a tendency to take the numbers as an ordered pair, and the only way to prevent this is to declare that you didn't mean to assign an order. Similarly, when you write the names for the numbers, unless you write them on separate sheets of paper, there is a tendency to interpret them as being given in order. We need to give the student a good example of an occasion where one is concerned with unordered pairs of numbers. Here it is. Throw two indistinguishable dice. From them you obtain two numbers. Which one is the first number? This is a silly question for you don't have an ordered pair. But, make one die red or mark it in some other way, and agree that from it you will obtain the first number, and you do get an ordered pair.

* * *

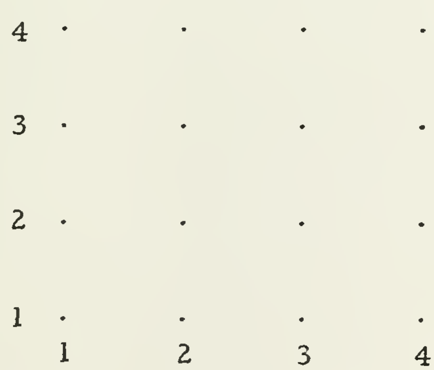
Notice in these exercises our use of words like 'give' and 'obtain' to get from dice to numbers. Actually, you look at a die and you see uppermost a name for a number (a different kind of name than those previously considered). When you recognize the name, you know which number is

(continued on T. C. 4B)

C. How many ordered pairs of numbers are there if there are three choices for first numbers and four choices for second numbers? Ten choices for first numbers and seven choices for second numbers? Twenty choices for first numbers and twenty choices for second numbers?

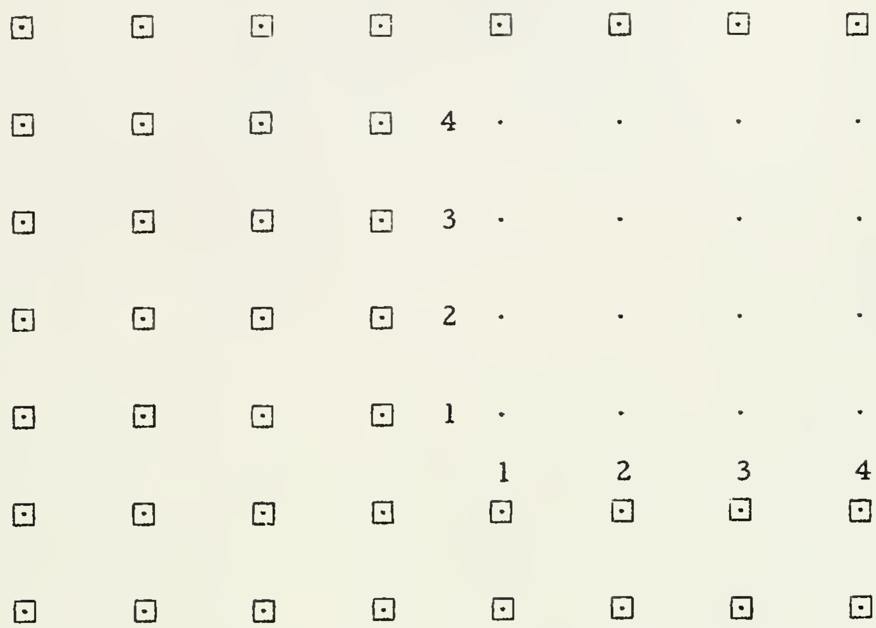
D. If you throw two dice, you obtain a pair of numbers. But such a pair is not an ordered pair because you do not know which number is the first number and which number is the second number. (Usually you don't care because you add the two numbers and addition is commutative. Explain.) You could use dice to obtain ordered pairs of numbers by having one of the dice red and one white and agreeing that the red one would give you the first number and the white one would give you the second number. How many different ordered pairs could you obtain from these two dice?

E. Suppose you had the problem of giving labels to buildings as above or labels to dots as in Part A, but from time to time more buildings or dots were added to the array. Each time an expansion is made you could renumber. The old buildings would get new names and you would have to learn them all over again. If, at first, you had this:

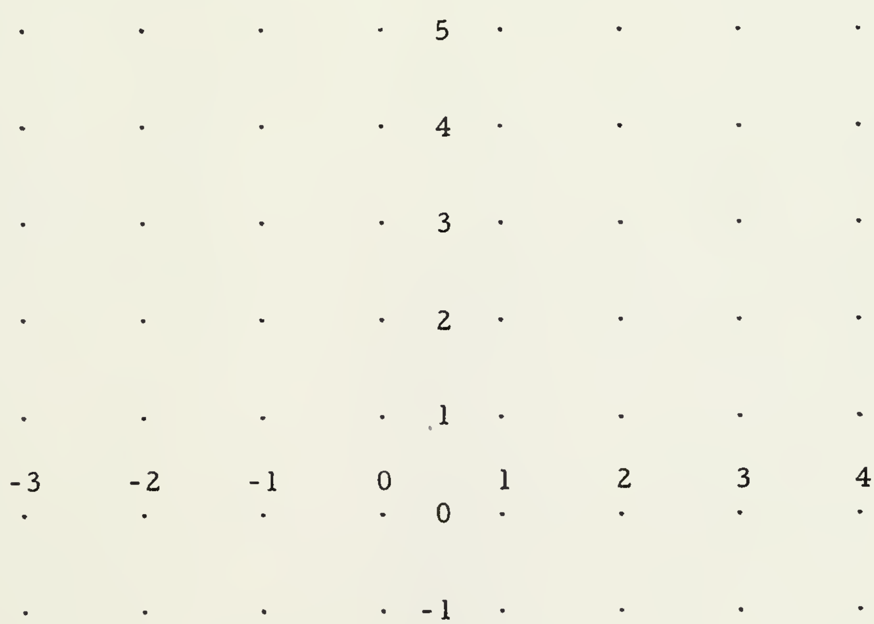


Teachers can obtain this paper in quantity from the UICSM. It is very useful in doing the exercises in Parts E and F, for it avoids having students waste time in making their own lattices. [You may want to ration the paper to keep students from exhausting the supply by excessive playing of DOTS (completing squares and putting in initials).]

and an expansion were made to this:



then a good way of extending your naming system would be this:



Then, the lower left point above corresponds to (-3, -1). Make a drawing like this for a further expansion with points corresponding to every ordered pair of numbers that there is with first number belonging to the set:

-3, -2, -1, 0, 1, 2, 3

There are six outcomes that are possible in this experiment. The student believes because of the general nature of the results of many throws and, perhaps just as importantly, because of his intuitive feelings about the symmetry of a die, that any of these six outcomes is equally likely. Here, 'equally likely' is a primitive term. Then, we can define the probability of any one of these equally likely events to be the reciprocal of the number of events. Thus, we are choosing a measuring scale where probability 1 is associated with an event which is "certain" to occur [such as getting 1 or 2 or 3 or 4 or 5 or 6 when throwing a die], and where probability 0 is associated with an event which is not among the possible outcomes of the experiment [such as getting 62 as the sum when throwing two dice]. In elementary probability problems of the type the student will be working here, the real problem is to get a clear picture of what are the underlying equally likely events to which probabilities can easily be assigned. This physical experiment is to give the students some accurate intuition.

8. $-3(r - s) - 6(2r + 3s) - (-r - 2s)$

9. $|6| - |9|$

10. $|8 - 13| \times (-2)$

11. Which of the following statements is an instance of the commutative principle for addition?

(a) $(3 + 7) + (12 + 9) = (3 + 7 + 12) + 9$

(b) $(12 + 3 + 7) + 9 = 12 + (3 + 7 + 9)$

(c) $(7 + 9) + (12 + 3) = (12 + 3) + (7 + 9)$

(d) $7 + (9 + 3) + 12 = 7 + 9 + (3 + 12)$

12. For every $h \neq 0$ and every $m \neq 0$, if one pound of apples costs $\frac{3}{h}$ cents then $\frac{5}{m}$ pounds of apples cost _______ cents.

* * *

You have been supplied with sheets labelled 'UICSM Lattice Paper-2-56' for your students to use in recording their results. These results can then be totalled on one of these sheets. It will be interesting to your students to compare the results of the class total with their individual totals, and with the summary sheet included in the Commentary for page 4-7.

* * *

After throwing the dice the student will feel intuitively that the chances of any one face of a die being uppermost are the same as the chances for any other face. Some students may already be able to express themselves quantitatively. For example, they may say "the odds are 1 out of 6," or "the odds are one chance in six," or "the chances are 1 in 6." Whatever they say should, upon reflection, mean something like this.

(continued on T. C. 6D)

give him a true statement from Exercise 10 or not. And, he can try every point in the lattice if necessary! Students will probably come up with the idea that the first condition leads to one "line" of points, and that the second condition leads to another "line" of points. To meet both conditions, a point must be in both sets. You can see the obvious connection with solving systems of linear equations, where one equation gives you one line and the other equation gives another line, and the point of intersection of the two lines is the solution of the system.

* * *

Part G may surprise you. You will be even more surprised by the kind of questions your students will be able to answer before they finish it. In a class of 30 students, have each student make and record say, 25 throws. (Students can work in pairs.) On the basis of our experience, we predict that you will probably have a few students who will make an astronomical number of throws. We think that students will be fascinated by the process of throwing a pair of dice and having the outcome determine a point.

* * *

For a quick quiz (class or take-home) to maintain some skills of earlier units, try this one.

Simplify.

1. $\frac{-15a}{-3}$

2. $m + n - 2m + 6n$

3. $(-3.2)(-5a)$

4. $56b \div (-4)$

5. $\frac{2}{3} \bigcirc + \frac{1}{6} \bigcirc$

6. $\frac{c}{21} \times \frac{3d}{4c}$, $[c \neq 0]$

7. $4(a - 3b) + 5(3a - b) - 7(-2a + 5b)$

(continued on T. C. 6C)

After students have prepared the drawing according to the instructions on page 4-5, ask them to examine the exercises of Part E and decide whether they will be able to locate points corresponding to each of the ordered pairs given. [They should recognize that points corresponding to the ordered pairs named in Exercises 9 and 15 cannot be located unless the lattice drawing is expanded.]

* * *

Part F contains the kind of problems we referred to in the Introduction. When the student tells you that the answer to Exercise 1 is '10 points', he has gone a long way toward understanding why the locus of '$y = 2$' is a line and not just a single point. (Naturally, you should not bring in "x-coordinate and y-coordinate" or any "x, y ideas" at this time.) In doing Exercise 5 he is making preparation for drawing the locus of the sentence:

$$x > 1 \quad \text{and} \quad y < 2.$$

Notice that another advantage of these finite lattices is that the student can actually count the points in a set to give his answer. <u>He should be encouraged to look for shortcuts in making these counts.</u> Notice, also, how clearly you can distinguish between $>$ and $\geq$. It makes a difference of a whole row of points.

* * *

Exercise 10 may be difficult for some students. Do not give your students any formal methods. In reading the two conditions, they may feel that something is wrong and that we must have intended this to be two problems. Point out that because of the 'and' they should hunt for points where <u>both</u> conditions are satisfied. If the student is baffled, remind him that if he picks a point he can decide whether numbers for that point

(continued on T. C. 6B)

and second number belonging to the set:

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4.$$

On your drawing locate the point which corresponds to:

1.	(2, 3)	2.	(3, 2)	3.	(-3, 2)
4.	(-3, -2)	5.	(-2, -3)	6.	(2, -3)
7.	(1, -5)	8.	(0, 0)	9.	(4, 0)
10.	(-3, -3)	11.	(-3, -5)	12.	(0, -5)
13.	(2 - 3, 4 + 0)	14.	$\left(\frac{8}{4}, \frac{-6}{-3}\right)$	15.	$\left(3 \div \frac{3}{5}, 2 \div 4\right)$

F. In your drawing for Part E, how many points are there corresponding to ordered pairs with:

1. first number 2
2. second number -3
3. first number greater than or equal to 2
4. second number less than or equal to -1
5. first number greater than 1 and second number less than 2
6. first number equal to second number
7. first number 1 larger than second number
8. second number twice first number
9. second number equal to 3 more than 2 times first number
10. second number 5 more than 2 times first number and with second number 3 more than first number

G. Obtain a pair of dice with one die red and one die white. (Any two colors will do, but you will have to interpret the instructions accordingly.) Mark out 36 dots in a rectangular array as you have been doing in the previous exercises with the dots placed one inch or more apart. Following the conventions you have used above, assign an ordered pair of numbers to each of the 36 dots. Choose the first numbers from the set:

$$1, 2, 3, 4, 5, 6$$

and choose second numbers from the same set. Agree that the

Total Results in the UICSM-FIRST COURSE Experiment With Dice (Unit 4), 1955-56

SECOND NUMBER	1	2	3	4	5	6	Row Totals	Percent of Total in Each Row
Percent of Total in Each Column	16.8	16.1	16.4	15.6	16.9	18.2		
Column Totals	1590	1526	1551	1480	1605	1723	Grand Total: 9475	
6	251	274	267	233	255	280	1560	16.5
5	236	293	270	248	283	291	1621	17.1
4	381	246	278	254	242	315	1716	18.1
3	240	240	230	254	261	267	1492	15.7
2	235	242	239	253	270	268	1507	15.9
1	247	231	267	238	294	302	1579	16.7
	1	2	3	4	5	6		

FIRST NUMBER

The most likely frequency for each ordered pair for 9475 throws is 263. Here is a chart showing the differences between 263 and the results obtained.

6	-12	+11	+4	-30	-8	+17
5	-27	+30	+7	-15	+20	+28
4	+118	-17	+15	-9	-21	+52
3	-23	-23	-33	-9	-2	+4
2	-28	-21	-24	-10	+7	+5
1	-16	-32	+4	-25	+31	+39
	1	2	3	4	5	6

By now, you and the class should be feeling the need for a neater language than say, "The chances are 5 in 18." Also, students should see that in problems like Exercise 3, and harder ones to come, these chances are "combined" to get other chances. So the chances act like numbers. In Exercise 5 you make the transition to probabilities, which are numbers between and including 0 and 1.

Students should have no trouble with this providing they actually carry out the instructions. They will find that there is only one point with a '12' beside it, a corner point. The probability of hitting this one point is $\frac{1}{36}$, and this gives the probability of getting 12 as sum. The student will find that there are six points with a '7' written beside each. The probability of getting 7 as the sum is $\frac{6}{36}$, or $\frac{1}{6}$. You may have to give a few more examples of how probabilities are found but your students will probably accept this without difficulty. Near the end of Exercise 5, we ask the student for the probability of "every outcome". He will probably give '1' as a response without even using his rule, but he should see that he can obtain this answer by formal means, also. He counts up all the points indicated and gets 36. Then he divides by 36 as he has been doing to find probabilities and gets 1. Similarly, he will laugh at the idea of getting 89 when throwing two dice. But he should see that of his 36 possible outcomes, 0 of them give him the sum of 89, and $\frac{0}{36} = 0$.

it is not easily adapted to probability. You will have to ask such a student to use the other method so that the entire class will talk the same language.

* * *

Now to specific exercises. In Exercise 1, the student should say something equivalent to 'the chances are one in six' as the answer to all five questions. In Exercise 2, he should give an answer equivalent to '1 in 36' for each of the six questions. In Exercise 3, the student should reason from his experience as follows.

> The chances of hitting (3, 2) by itself are 1 in 36.
> The chances of hitting (1, 5) by itself are 1 in 36.
> Then [intuition tells me], the chances of getting either (3, 2) <u>or</u> (1, 5) are 2 in 36 which is the same as 1 in 18.

Don't give the student any rule here; let him figure it out himself. He should quickly come to see that the chances of getting one of two points are 1 in 18. He probably has in the back of his mind the idea that if he were asked the chances of landing in any given <u>subset</u> of these 36 points he would count the number of points in the subset and then say that the chances were that number in 36. He can test this presently unverbalized rule on the last question in Exercise 3 because he has already agreed in Exercise 1 that the chances of getting into a given row (or a given column) are 1 in 6.

In Exercise 4, the student should be able to formulate by himself the correct rule:

> Count the points in the set in question. The chances are this number in 36.

(continued on T. C. 7D)

have gone far enough. You can say that the ratio of heads to tails will, in some sense, approach 1. But to make accurate the phrase 'in some sense' takes a great deal of technical machinery. One thing that this does not mean is that if you have been tossing coins for a while and the heads have a large lead on the tails, then the chances that you will get a tail are greater than before. Students often advance such an erroneous theory under the ignominy, "the law of averages". If a student believes in this, carry out the following experiment. Toss a coin in groups of three tosses. If the outcome of a group of three tosses is anything other than three heads, ignore it (you'll be ignoring about seven-eighths of your groups of three). But if the outcome is three heads, make another single toss and record its outcome. Do this many times. It will be apparent that tossing a coin after three heads in a row have been tossed does not bias the outcome of the toss.

By now you may be thinking that if probability is as tricky as we claim, then why teach it at all. It isn't tricky or difficult as long as 'probability' is left as a word without further definition or interpretation, but which one comes to understand by experience and intuition. You all know the wide applications of probability to nearly all kinds of organized research; this is one reason for teaching it. Another is that it is interesting to students at this age.

One other detail. You may find a student who prefers to say that the odds, or chances of getting, say, 3, when throwing one die are one to five. He is considering the six equally likely events and saying that one of them gives the outcome 3 and that five of them do not give this outcome. This is a perfectly good way to talk about odds, but

(continued on T. C. 7C)

Before going into the details of the exercises, a word of warning. Be very careful in this work to avoid trying to say what 'probability' means. Keep the discussions centered around the simple questions being asked and the finding of probabilities rather than their interpretation. For example, if you flip a balanced coin, it is usually assumed that "getting heads" and "getting tails" are equally likely events, and therefore that the probability of either of these events is $\frac{1}{2}$. Now, what does this last sentence tell you? If you answer, "the odds are even," or "there's as good a chance of getting one as there is the other," or "you are just as likely to get one as the other," you are safe because you have said no more than what was intended by the sentence. But if you say, "You'll get just as many heads as you will tails", you are wrong. The probability of getting exactly 5 heads in 10 tosses of a coin is small. [If you should run into someone who doesn't believe this, have him make tosses with a coin, ten tosses at a time, and record the total number of ten tosses made and also the number of ten tosses which came out five heads and five tails.] Even if you say "in the long run the number of heads and the number of tails will get closer and closer together" you will probably be wrong because if you actually make tosses it is highly likely that the <u>difference</u> between the number of heads and the number of tails will increase in the long run. [Notice how vague expressions like 'highly likely' and 'in the long run' creep into the discussion.] Now, if you said that the quotient of the number of heads by the number of tails will get "closer and closer" to 1, you would be getting warm, but you would still be wrong. It is quite possible that after 100 tosses the ratio is, say, 0.97, and after 200 tosses the ratio is, say, 0.95. Again, to talk about events such as these you would go back to discussing <u>their</u> probability. Well, one can go on and on setting up straw dummies and knocking them over like this, but we

(continued on T. C. 7B)

red die will give you the first number and the white one will give you the second number of an ordered pair of numbers. Thus, each time you throw your dice the result of the throw gives you one of the points on your diagram. Now, throw the dice. Make a small tally mark next to the point given by the throw. Repeat this process a few hundred times. (This could be a class project with each person making and recording, say, 25 throws and combining all the results.)

1. From your experience, what seem to be "the chances" that you will get a point in the third row when you make a throw? First row? Fourth row? First column? Third column?

2. What seem to be "the chances" that you will hit the point corresponding to (2, 3) when you make a throw? (4, 2)? (1, 5)? (2, 4)? (3, 3)? Any one point?

3. What are the chances of hitting in one throw either (3, 2) or (1, 5)? Either (1, 6) or (6, 1)? Either (1, 2) or (1, 3) or (1, 4)? Either (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), or (2, 6)?

4. Pick out any set of the points on your diagram. Give a method for telling the chances that you will land in this set in one throw?

5. Next to each of the points in your diagram write a numeral for the number you obtain when you add the first number and the second number in its corresponding ordered pair of numbers. How many points are there for which the sum is 12? What are the chances of "getting 12" when you throw two dice? How many points are there for which the sum is 7? What are the chances of "getting 7" when you throw two dice?

   Make a table showing the chances of getting all the possible sums for the two dice. Indicate the chances by giving the <u>probability</u> of getting each sum. You obtain the probability

(continued on next page)

we have come to see that some considerations of probability will add interest and meaning to certain related topics and therefore, we shall call on it mostly as a game whenever it seems appropriate.

* * *

In Section 4.02, use all these terms frequently in class. Your students will probably prefer 'first component' and 'second component' to 'abscissa' and 'ordinate', and so do we. We wish that it were not necessary for high school students to learn such antiquated terms and perhaps, before long, it will not be necessary. In the meantime, our students have to be able to understand most of the language on standardized tests; hence the inclusion of 'ordinate' and 'abscissa'.

Here is another example. In throwing one coin and one die, the probability

(A) of getting the pair (h, 1) is $\frac{1}{12}$,

(B) of getting the pair (h, 4) is $\frac{1}{12}$, and

(C) of getting a tail without regard for the die is $\frac{1}{2}$.

Every pair, A and B, A and C, B and C, of these events is a pair of mutually exclusive events. Therefore, we can conclude that the probability

of getting tails on the coin

or

of getting heads on the coin and 1 on the die

or

of getting heads on the coin and 4 on the die

is $\frac{1}{2} + \frac{1}{12} + \frac{1}{12}$, or $\frac{2}{3}$.

You can check this result by going back to the twelve primitive, equally likely events and seeing that eight of them give you the probability of one of these three events.

* * *

From time to time, we shall introduce more probability notions when we feel that they fit in naturally and when we feel that they will be of interest to your students. We have been quite hesitant to plan for a full unit of probability or statistics anywhere in the four year sequence because we have doubted whether enough could be accomplished with sufficient care to justify it. Recently, however,

(continued on T. C. 8G)

feeling the necessity to specify it, that no two of these events can occur simultaneously. Such a pair of events, where the occurrence of one of the events precludes the occurrence of the other event, is called a pair of mutually exclusive events. Let us look at an example of events which are not mutually exclusive. Consider throwing a die and the event "getting an even number". This event has probability $\frac{1}{2}$ (remember, we are considering a single throw of one die). Let us call this 'event A' or, simply, 'A'. Another event is "getting a number different from 4". This event has probability $\frac{5}{6}$. Let us call that event 'B'. Now, we see that A and B are not mutually exclusive because it is possible for both of them to occur simultaneously. For example, if you throw a die and obtain 2, then both event A and event B have occurred. Another way of saying that two events are mutually exclusive is that the probability of both of them occurring is 0. You can see that the probability of both A and B in this case is $\frac{2}{6}$, or $\frac{1}{3}$, and not 0.

The importance of mutually exclusive events is that their probabilities "add up". The student has been using this rule intuitively throughout these exercises. For example, he assumed that since the probability with two dice of (2, 4) is $\frac{1}{36}$ and since the probability of (1, 3) is $\frac{1}{36}$, the probability of either (2, 4) or (1, 3) is $\frac{1}{36} + \frac{1}{36}$, or $\frac{1}{18}$. We get correct results by adding the probabilities because the probability of getting in a single throw of two dice both (2, 4) and (1, 3) is 0. You know the probability of both events is 0 because such an outcome is simply impossible.

(continued on T. C. 8F)

whether you get 6 on the red die or not. We say that "getting 2 on the white die" and "getting 6 on the red die" are <u>independent</u> <u>events</u>. The student has learned enough about the probabilities with dice to assert that the probability of each of these two independent events is $\frac{1}{6}$ and the probability of the occurrence of both of these events together is $\frac{1}{36}$. Here the student has an illustration of the rule that the probability of <u>both</u> of two independent events is the <u>product</u> of the probabilities of each of the events separately. For example, if you throw a coin and a die together, the probability of getting heads on the coin is $\frac{1}{2}$ and the probability of getting 1 on the die is $\frac{1}{6}$. These two events are independent. [We decide this intuitively because we <u>just</u> <u>know</u> that the outcome for the coin does not affect and is not affected by the outcome for the die.] Following the rule, we find that the probability of getting heads on the coin <u>and</u> 1 on the die is $\frac{1}{2} \times \frac{1}{6}$, or $\frac{1}{12}$. You can see that this kind of analysis gives more rapid results than the method the student has been using where he would say that the equally likely events are:

(h, 1), (h, 2), (h, 3), (h, 4), (h, 5), (h, 6)
(t, 1), (t, 2), (t, 3), (t, 4), (t, 5), (t, 6),

where 'h' stands for heads, 't' stands for tails, and the numerals stand for the outcome for the die. Since there are 12 of these equally likely events, the student assigns probability $\frac{1}{12}$ to each of them. He finds that only one of these equally likely events gave him the outcome "heads on the coin, 1 on the die", and he comes to the same conclusion as he did when using the rule.

Here is another fundamental idea. If you consider throwing one die, there are six primitive events which are equally likely and therefore are assigned probability $\frac{1}{6}$. Moreover, you know, without even

(continued on T. C. 8E)

these dice-throwing exercises. They provide a painless way for students to get practice in identifying abscissas and ordinates of points in the coordinate plane. They gain a deep appreciation for the notion of ordered pair since they recognize that the order of an element in the pair is given by the color of the die which corresponds to the element. This practice in recognizing order complements nicely the procedure used in determining order from the written symbols for ordered pairs. We have found that last year's UICSM students rarely made a mistake concerning the order of components.

If you do use the dice-throwing exercises in your conventional classes, please let us know the outcome.

* * *

If you should find that your class is particularly interested in these probability problems, there are several ways in which you can go deeper into the subject.

One way, of course, is to ask for more complicated probabilities in Exercise 7. The student can give the probability of any sum if he is patient and meticulous enough to find all of the points in his $6 \times 6 \times 6$ cube which give him that sum. Therefore, the real quest should be for shortcuts in locating points with a certain sum.

Another direction in which you can go more deeply into this work on probability is an informal discovery of some of the elementary rules for combining probabilities. If you throw a red die and a white die together, the number which you obtain from one of the dice is not affected by the number which you obtain from the other. Thus, for example, it is just as likely that you will get 2 on the white die

(continued on T. C. 8D)

He organizes his trials by first looking for triples with first number 1, and then looking for triples with first number 2, etc. Since there are 6 points, the probability of getting the sum 5 is $\frac{6}{216}$ or $\frac{1}{36}$. Again, the student should see that he has all the machinery he needs to tell the probability of obtaining any sum, but the job of counting up points would become laborious for the sum, say, 9. Urge him to find shortcuts for counting the points in sets like these.

* * *

Here are other questions you might use in considering the outcomes of a throw of two differently colored dice.

1. What are "the chances" that you will get an ordered pair with components whose
   a) sum is 5?
   b) sum is 7 or 11?
   c) sum is a number greater than 7 but less than 11?
   d) sum is a number ≥ 2 and < 12?

2. What answers would you give to the questions of Exercise 1 above if you threw two dice of the <u>same</u> color?

* * *

Teachers reported that our "dice-throwing" exercises were extremely popular. In fact, students in other classes wanted to engage in the same activity. Although we are somewhat jealous of this exercise because it does create a certain atrraction to the UICSM program, we would not mind your giving Part G to "conventional" classes. [You can get extra sheets of the necessary lattice paper by writing to us.] Aside from the understandings students will derive as far as probability is concerned, there is another pedagogical value in

(continued on T. C. 8C)

In Exercise 6, he should find a set of 16 of the 36 points. Thus, the probability is $\frac{16}{36}$ or $\frac{4}{9}$. The student should realize now that he can find the probability of any event involving one throw of two dice. These exercises can be extended with questions such as "What is the probability of getting a sum which is an even number?".

The seventh exercise is a test of the students' ability to generalize. Students will probably get solutions by a variety of correct methods. Here is the method we would like the class ultimately to settle on. If the three dice were colored, a throw would give an ordered triple of numbers. There are $6 \times 6 \times 6$, or 216 such ordered triples and they are all equally likely events. Graphically, think of a cube array of lattice points with six points on an edge. A probability of $\frac{1}{216}$ is assigned to each of these 216 points. Now we are ready to answer questions. First question: What is the probability of getting the sum 3? Answer: There is only one point in the cube which gives the sum 3. It is the corner point corresponding to (1, 1, 1). Since there is only one such point, the probability of getting the sum 3 is $\frac{1}{216}$. Second question: What is the probability of getting the sum 18? Answer: Again, there is only one point which gives a sum of 18. It is the corner point (diagonally across from the point considered previously) corresponding to (6, 6, 6). Since there is only one point, the probability is again $\frac{1}{216}$. In the last question the student should realize that the whole thing amounts to finding how many points in the cube array give sum 5 and that this amounts to finding how many ordered triples there are the sum of whose components is 5. By trial, we find:

(1, 1, 3),	(2, 1, 2),	and	(3, 1, 1).
(1, 2, 2),	(2, 2, 1),		
(1, 3, 1),			

(continued on T. C. 8B)

like this. If the chances are say, "6 out of 36", then the probability is the number $\frac{6}{36}$, and sometimes it is more convenient to approximate such a fractional number by a decimal, in this case, 0.1667. Add up all the probabilities in your table. (You should have 11 probabilities.) What is the probability of getting either the sum 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 when you throw two dice? What is the probability of getting 89 when you throw two dice?

6. Find the set of points in your diagram where the sum is either 6, 7, or 8. What is the probability of landing in this set when you throw two dice? What is the probability of getting either the sum 6, 7, or 8 when you throw two dice?

7. If you were to throw three dice at once, what is the probability that you would obtain the sum 3? 18? 5?

4.02 Plane lattices.--The rectangular arrays of points with which you worked in the preceding section are called plane lattices. Each point in a plane lattice corresponds to an ordered pair of numbers; the numbers in the pair are the coordinates of the point. The first number in the ordered pair is the first coordinate or the abscissa of the point corresponding to the ordered pair; the second number is the second coordinate or the ordinate of the point. The point is called the graph of the ordered pair of numbers.

Then consider these possibilities:

1. Find all the points such that $f = 1 + s$,

<u>or</u>

Find all the points such that $f + s = 2$.

2. Find all the points such that $f = 1 + s$ <u>or</u> $f + s = 2$.

3. Find all the points such that $f = 1 + s$,

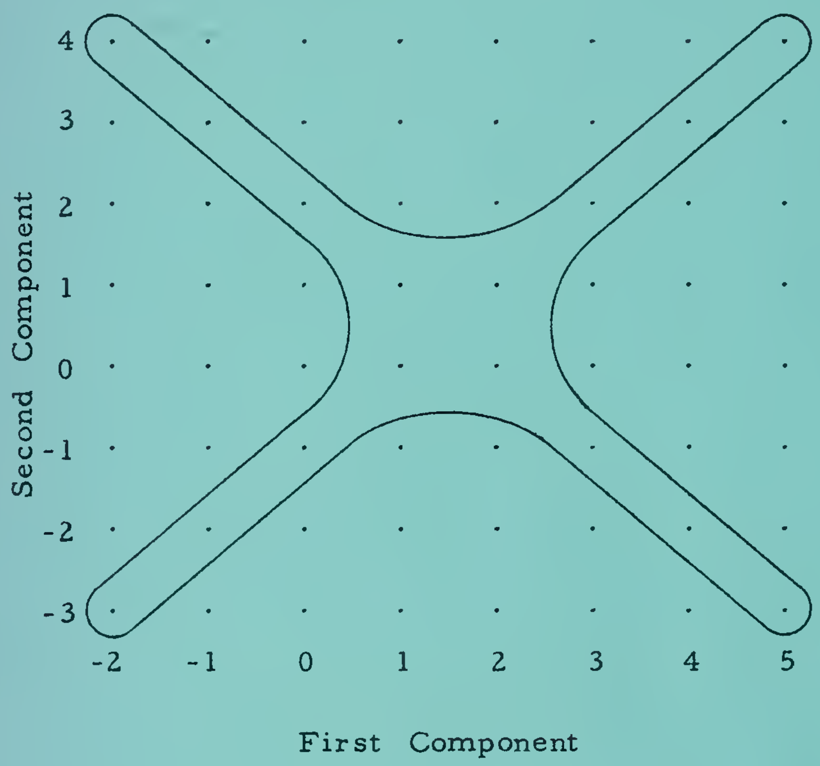
<u>and</u>

Find all the points such that $f + s = 2$.

4. Find all the points such that $f = 1 + s$ <u>and</u> $f + s = 2$.

Get the class to recognize the different implications of these statements.

After discussing the exercises of page 4-9, you may want to experiment with this. Before class, draw on the board this figure:



Ask the class, "What expression, or expressions, could you write which would serve as a "rule" for finding all the points contained in the cross marked on the lattice?"

Some will probably suggest (using 'f' for 'first component' and 's' for 'second component'):

$$f = 1 + s \quad \underline{\text{or}} \quad f + s = 2.$$

Then you will need to consider with the class whether it is correct to use 'or'. Ask whether 'and' is a better word to use.

(continued on T. C. 9C)

Exercises for a quick review quiz.

Find all the elements in each set.

1. $\{x: \ x + 9 = 33\}$

2. $\{a: \ 3a + 12 = -24\}$

3. $\{d: \ \frac{d}{2.4} = 3\}$

4. $\{m: \ -5 + m = m - 5\}$

5. $\{s: \ 51 = 3s - 9\}$

6. $\{y: \ 25\% y + 1 = -15\% y + 21\}$

7. $\{x: \ 4x + 7 = -x + 14 + 5x\}$

8. $\{r: \ \frac{13r - 13}{6} = \frac{39}{6}\}$

9. $\{c: \ \frac{3c + 42}{8} = 0\}$

10. $\{w: \ \frac{-w}{6} - \frac{w + 2.5}{3} = \frac{w}{12}\}$

11. The formula '$C = \frac{5}{9}(F - 32)$' is used in converting from degrees Fahrenheit to degrees Centigrade. What is the temperature in degrees C corresponding to 42 degrees F?

12. For which of the following pairs are the solution sets of the equations the same?

a) $bb = 36$
$b = 6$

b) $7n + n = 0$
$7n = 0$

c) $a = a$
$\frac{a}{a} = 1$

d) $r + 2 = 3$
$3r + 2 = 6$

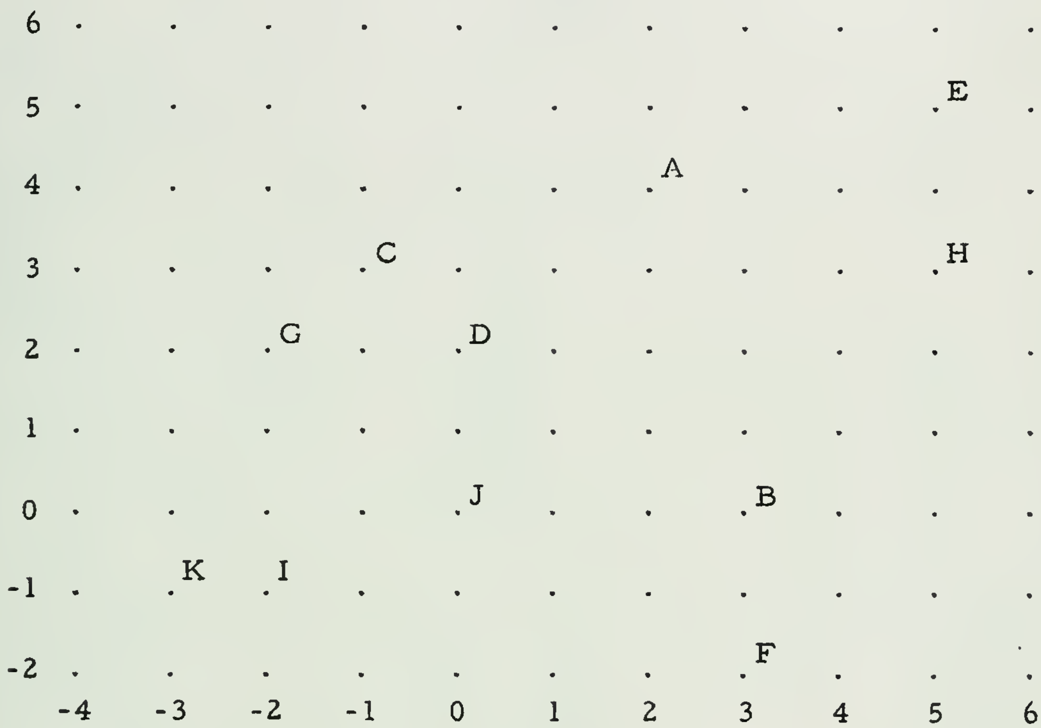
* * *

The questions here are to fix 'abscissa' and 'ordinate' in the student's mind. We want also to bring the students back from the probability excursion to a plane lattice with both positive and negative components of points in preparation for the extension to an unbounded lattice plane on page 1-10.

* * *

(continued on T. C. 9B)

Study the following diagram of a plane lattice and answer the questions which follow the diagram.



(1) Give the ordered pairs of numbers which correspond to the eleven labeled points in the diagram above.

Sample. A: (2, 4)

(2) Give the abscissa of each of the labeled points.

(3) Give the ordinate of each of the labeled points.

(4) How many points in the lattice have equal first and second coordinates?

(5) Label with 'L' the point corresponding to the ordered pair (4, 3).

(6) Label with 'M' the point whose abscissa is 3 and whose ordinate is 4.

(continued on next page)

In answer to questions (7) and (8) most of your students will naturally draw "straight" lines. Of course, the part of the lines which are between the points are not in the lattice plane. Therefore, it is actually acceptable if the student purposely draws dotted, wavy lines through these points. Any student who recognizes this possibility is demonstrating unusual understanding.

* * *

There is an interesting side road for enrichment here, and for preparation for SECOND COURSE. The student by now has an idea of a "straight line" in a lattice plane. He can make the idea rigorous by deciding that a "straight line" is a set of ordered pairs of numbers in which each second component can be obtained from the first component by multiplying by a previously selected number, and adding to this product another previously selected number. It is interesting to check, either intuitively or with algebraic rigor, various properties of these "straight lines". For example, if you define two "straight lines" to be parallel when they have no point in common, is it the case that if two lines are parallel to a third line, they are parallel to each other? Must two "straight lines" intersect either in no points, one point, or every point? Is it true that on such a line, between any two points there is another point? Is it true that given two points on a "straight line", there is always another point on the line which is not between the given two? You can probably think of many more properties to test. It is interesting to see that a considerable number of the properties of a straight line in a "complete" number plane are possessed by these "straight lines" in a lattice plane. [See SECOND COURSE, 1957-58, pages 1-72 through 1-75.]

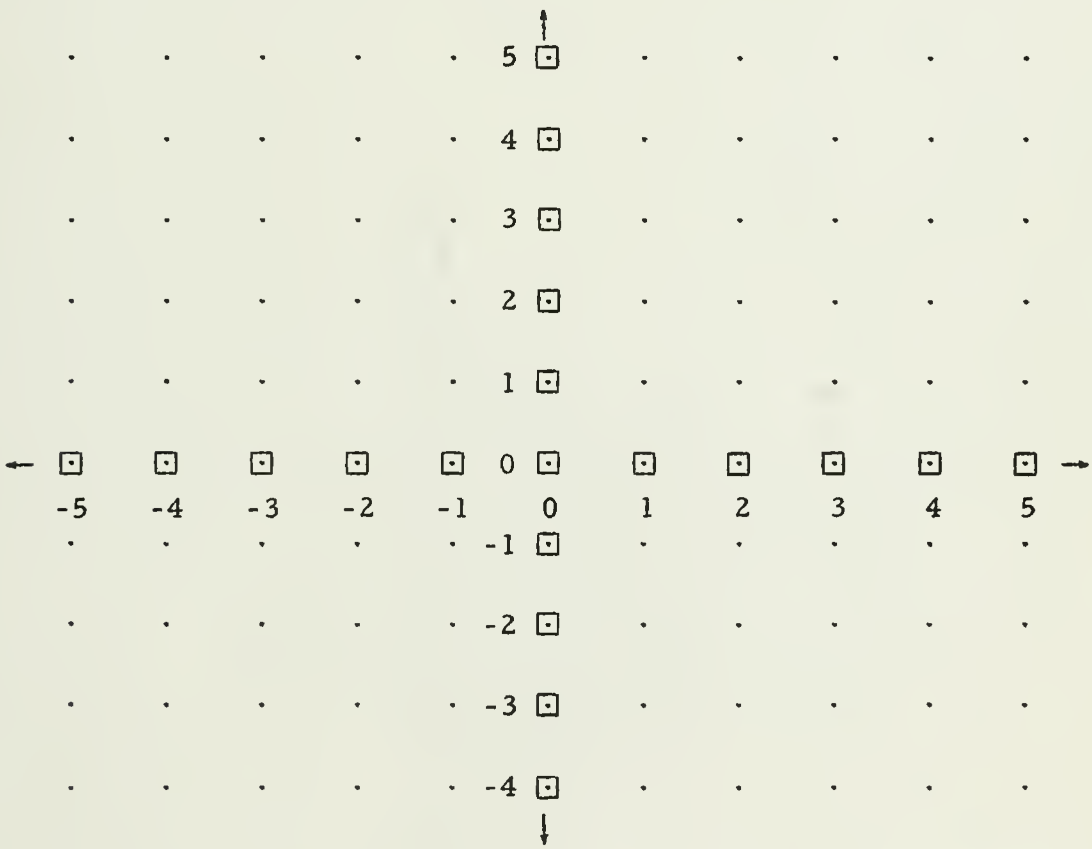
(7) Draw a dotted line which connects all the points with abscissa 0.

(8) Draw a dotted line which connects all the points with ordinate 0.

(9) Label with 'W', 'X', 'Y', and 'Z', respectively, the graphs of the ordered pairs (0, -1), (-2, 0), (-2, -2), and (6, 6).

LATTICES WITH INDEFINITELY MANY POINTS

Imagine a plane lattice with so many points that given any ordered pair of whole numbers (negative, zero, or positive), you could find a point corresponding to this pair, even (1, 263, 001, -26, 982). Obviously, it is not possible to draw a diagram of the entire lattice. Nor is it possible to write a numeral for every row and for every column. Instead, we draw a diagram of part of the lattice and mark those points each of which has 0 as one of its coordinates.



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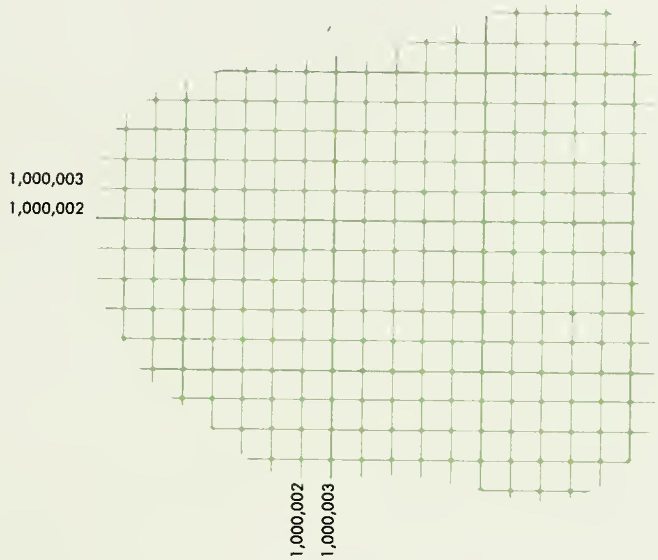
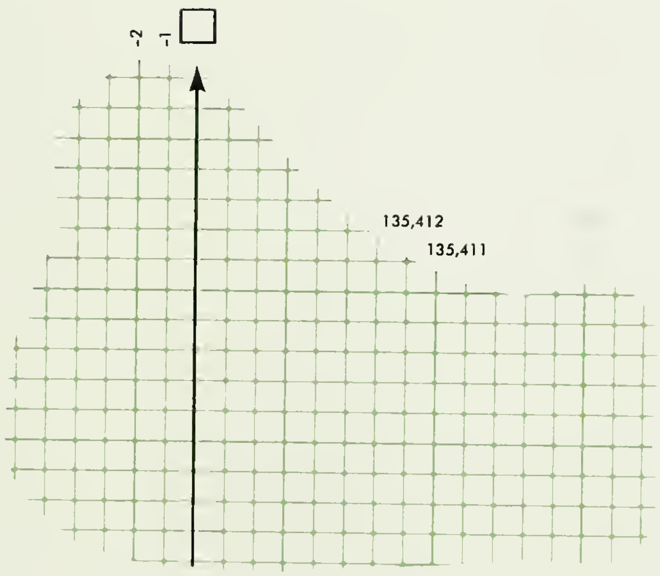
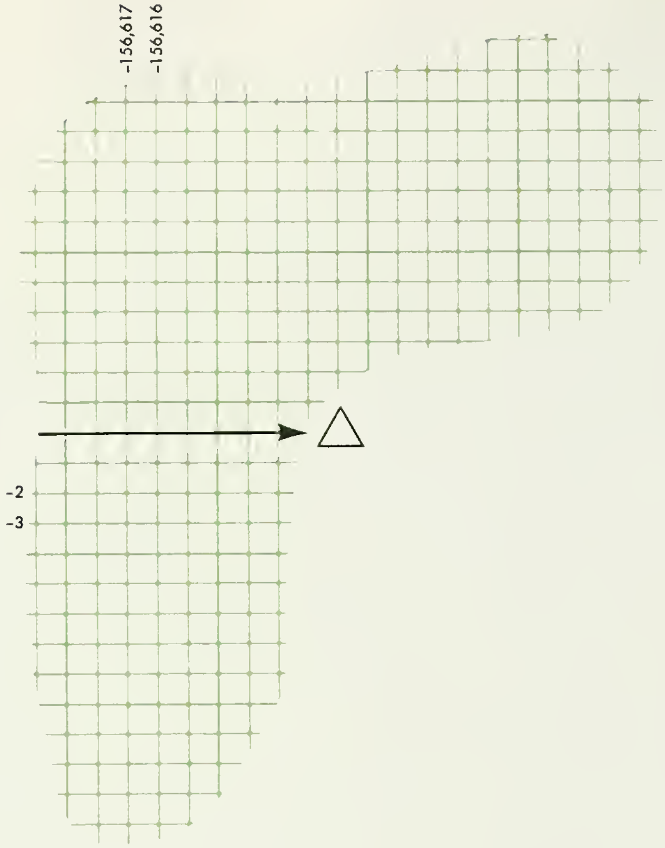
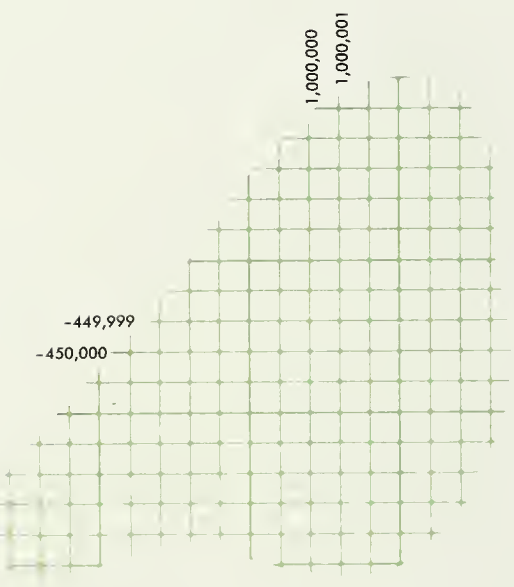




COORDINATE PLANE PAPER C_1

NAME ______________________________





COORDINATE PLANE PAPER C_2

In the above exercises some points whose graphs do not fall in any of the regions were <u>intentionally</u> included. [The commas have been omitted from the numerals in these exercises because they could be confused with the commas that separate the names for the two components of the ordered pair.] Here are answers to the above exercises.

A. C_1-u	B. C_2-u	C. C_1-d
D. C_2-d	E. C_1-u	F. C_2-u
G. C_1-d	H. Not in any region	

I. Not in any region [Compare with point C]

J. Not in any region

* * *

Throughout this material, we are postponing the discussion of an important issue. If you have thirty students in class, they will have in front of them thirty sheets of lattice paper, and certainly a point on one of these sheets of paper is different from any point on any other of the sheets of paper. So, clearly, when we say <u>the</u> point corresponding to (3, 2), we are not talking about points on paper. Further support of this is found in the fact that points on paper have actual size, and we think of points as merely positions with no internal area or structure. But if points are not marks on paper, what are they? This question is faced squarely in SECOND COURSE. The word 'point' [also 'straight line'] is regarded as a primitive, or undefined, term in the deductive theory of geometry. In each <u>model</u> of the deductive theory, the word 'point' is defined precisely. In the number plane model, a point is an ordered pair of directed numbers, and a straight line is the solution set of a linear equation. In the present unit, the students are becoming acquainted with the number plane model. Hence, it is appropriate to let students call the ordered pairs of directed numbers 'points' in preparation for a definition they will encounter in SECOND COURSE.

You should treat carefully in class the first several sentences. [Substitute 'component' for 'coordinate'.] Compare the sentence 'Latitudes run east and west, but measure distances north and south' with 'The first component axis is the set of points with second component 0'.

* * *

The student should have little difficulty in making this extension to the lattice plane with indefinitely many points. To be sure that the students fully grasp this idea, it will be instructive to distribute sheets of UICSM Coordinate Paper C_1 and C_2. [Students spend so much time working near the origin that they sometimes fail to realize that a coordinate of 1,000,000 is also possible.] You may want to spend 15 or 20 minutes discussing this paper and plotting points on it. Here are some exercises taken from the 1956-57 edition of SECOND COURSE which can be carried out with this paper.

Plot each of the given points and tell which region it is in. [The four regions are called 'C_1-u', 'C_1-d', 'C_2-u', and 'C_2-d'.]

A. (1 000 003, 1 000 003) B. (-156 616, -2)

C. (5, 135 410) D. (999 999, -450 003)

E. (1 000 008, 999 998) F. (-156 620, 0)

G. (0, 135 406) H. (156 602, 4)

I. (135 411, -2) J. (0, 0)

[We think students will notice, as they examine the C_1 and C_2 paper, that the arrows indicate the positive direction, and that the axis labelled with a 'Δ' is to be considered the first component axis.]

(continued on T. C. 11B)

The set of points each of which has second coordinate 0 is called the first coordinate axis. Points in this set are lined up horizontally in the diagram. The set of points each of which has first coordinate 0 is called the second coordinate axis. The points in this set are lined up vertically in the diagram. Note that the two sets of points have one point in common. This point corresponds to the pair (0, 0) and is called the origin.

The origin and the coordinate axes are useful in locating graphs of ordered pairs. For example, suppose you are trying to find the graph of (3, 4). First, put your finger on the origin. Then, move along the first coordinate axis until you locate the point corresponding to (3, 0). You are in the column which contains, also, the graph of every ordered pair with first number 3. Therefore, this column contains the graph of (3, 4). Now, return to the origin and move your finger along the second coordinate axis until you locate the point corresponding to (0, 4). You are in the row which contains, also, the graph of every ordered pair with second number 4. Therefore, this row contains the graph of (3, 4). So the graph of (3, 4) belongs to two sets of points--the set of points in the third column to the right of the origin, and the set of points in the fourth row above the origin. It is now easy to locate this point.

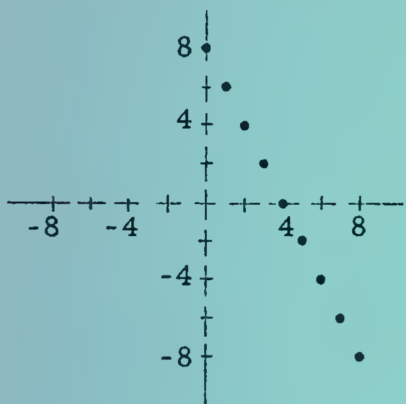
The process of locating a point when you are given its coordinates is often called plotting a point.

EXERCISES

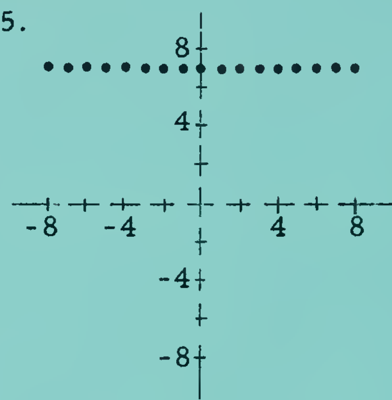
[Note: To do the exercises which follow you will need to make drawings of lattices in which you label the coordinate axes. You can simplify the drawing task by providing yourself with cross section paper and placing dots on the intersections of the printed lines.]

In Exercise 3 of Part B be sure students realize that the "easy" points (0, 9) and (9, 0) are not allowed. They are not in the lattice with which the student is working.

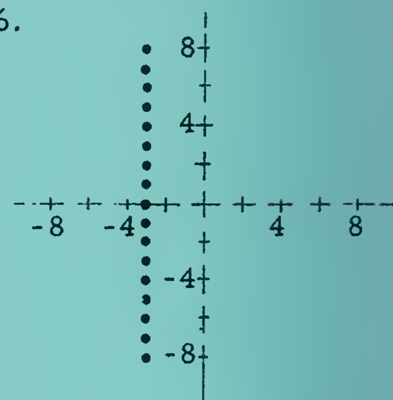
4.



5.



6.



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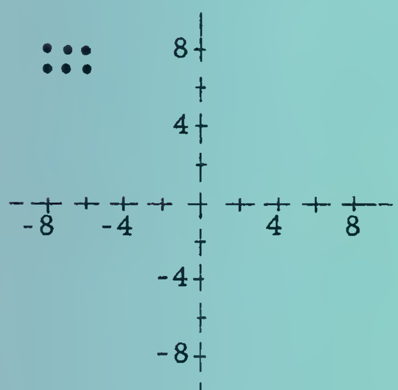

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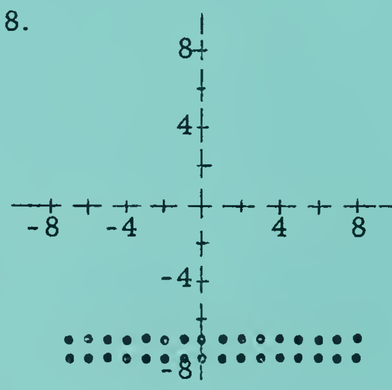

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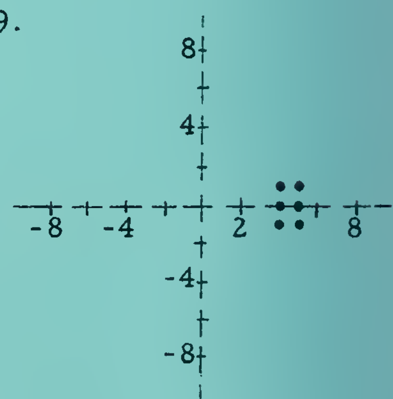

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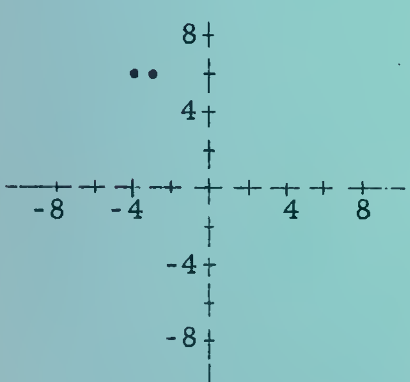


The students will require several sheets of lattice paper for Part B in order to keep the graphing manageable. Perhaps three exercises per sheet will be a good allocation. Students can mark points in the same locus by encircling each point or putting a tiny square or triangle around each point, or by using pencils having colored lead. Now is the time to develop in the students good habits concerning the labelling of their diagrams.

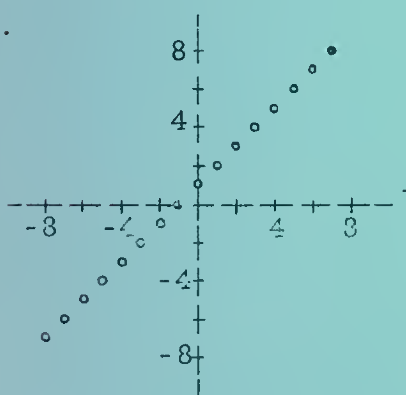
* * *

We want to report the sad tale of the UICSM teacher who underestimated his students. When Mr. Dietz checked the homework answers aloud for Part B in class (but before he had asked students for their scores), he told the students that anyone who had all ten exercises correct could claim a triple-dip ice cream cone. Unless you are prepared to buy eight of these ice cream cones (as Mr. Dietz had to do), do not challenge your students in a similar way!

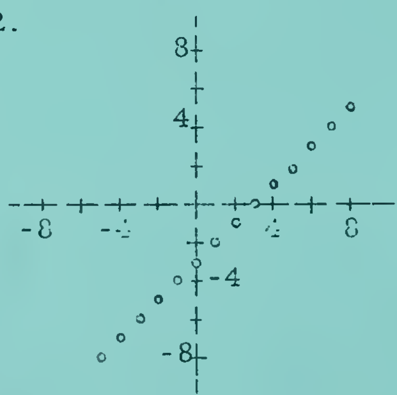
* * *

Here are rough sketches for the exercises of Part B.

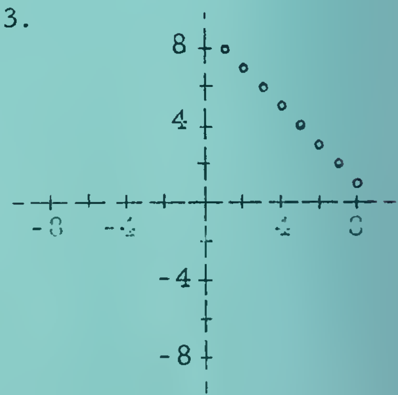
1.



2.



3.



(continued on T. C. 12B)

A. Plot the points whose coordinates are given by the ordered pairs. Label each point with the given letter.

A: (3, 5)	B: (2, 5)	C: (-3, 1)	D: (2, 0)
E: (0, 0)	F: (0, -2)	G: (6, 1)	H: (1, 6)
I: (-3, -4)	J: (4, -3)	K: (10, -10)	L: (-9, -9)

B. Draw a diagram of a lattice. Your diagram should contain enough points so that for every ordered pair of whole numbers (x, y), if $-8 \leq x \leq 8$ and if $-8 \leq y \leq 8$, then there is a point which is the graph of (x, y). [In your diagram how many points are there in the first coordinate axis? In the second coordinate axis?] Plot sets of points according to the following instructions. Indicate the points in each set by marking them in some particular fashion.

1. The set of all points with first coordinate equal to 1 less than second coordinate.
2. The set of all points with abscissa 3 more than ordinate.
3. The set of all points such that the sum of the coordinates of each point is 9.
4. The set of all points such that for each point 8 is the sum of the ordinate and twice the abscissa.
5. The set of all points with ordinate 7.
6. The set of all points with abscissa -3.
7. The set of all points with first coordinate less than -5 and with second coordinate greater than 6.
8. The set of all points with ordinate less than -6.
9. The set of all points with ordinate less than 2 but greater than -2 and with abscissa greater than 3 but less than 6. How many points are there in this set?
10. The set of all points with abscissa greater than -5 but less than -2 and with ordinate 6.

Part C contains a bit of formal work with set operations. Students should know well the words 'intersection' and 'union'. Ideas about sets permeate the UICSM program. In order to derive maximum benefit from "set thinking", students need to start early. [We think the elementary school is a good place to begin.] As documentation in support of our work with sets, we quote from the Introduction to UNIVERSAL MATHEMATICS, Part II, "Structures in Sets" (Committee on the Undergraduate Program of the Mathematical Association of America, 1955):

> "The general trend of development of civilization continues to require men to do more and more thinking in terms of sets."

[If you should read in older books about sets, you would find 'sum' and 'join' used as synonyms for 'union'. Also you would find 'meet' and 'product' for 'intersection'.]

* * *

At this point you should introduce the symbols used to indicate the intersection and the union of sets:

> In order to save space and make ideas easier to discuss, it is customary to abbreviate 'the intersection of Set I and Set II' as 'I $\cap$ II', and to abbreviate 'the union of Set I and Set II' as 'I $\cup$ II'.

C. Below are four diagrams of the same part of a plane lattice. Each diagram shows a set.

Set I is the set of all points with abscissa greater than 0 but less than 4 and ordinate greater than 0 but less than 4. There are 9 points in that set.

Set II is the set of all points with abscissa greater than 1 but less than 5 and ordinate greater than 1 but less than 5. There are 9 points in that set.

The third diagram (lower left) shows all the points which belong to both set I and set II. This set of points is called the intersection of sets I and II. There are 4 points in the intersection of sets I and II. [The intersection of two sets of points is the set of all points which belong to both of the two given sets.]

The fourth diagram shows all the points which belong either to set I or set II. This set of points is called the union of sets I and II. There are 14 points in the union of sets I and II. [The union of two sets of points is the set of all points which belong to either or both of the two given sets.]

Some of your students may want to say that I $\cup$ II contains 18 points because "you should count some of the points twice". If they do this, they are confusing the

union of two sets

with the

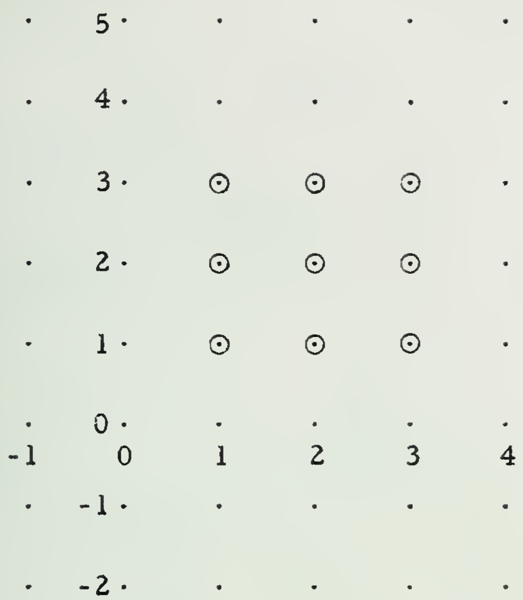
sum of two numbers.

These are very different ideas. A particular point cannot "belong to a set twice". It either belongs or it does not belong.

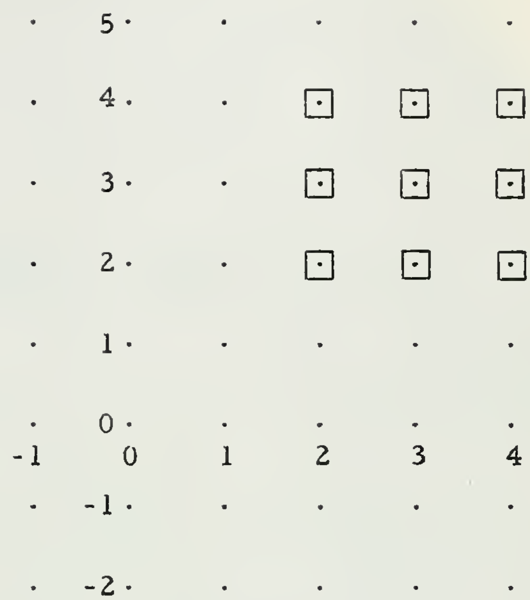
A playful teaching method we have used runs like this:

> Go to each point and ask it, "Do you belong to either of these two sets?" If it says 'yes', take it; if it says 'no', don't take it. When you have asked every point, you will have the union of the two sets in question. <u>If you ask a point the question several times, you are doing extra work but it won't change the result</u>.

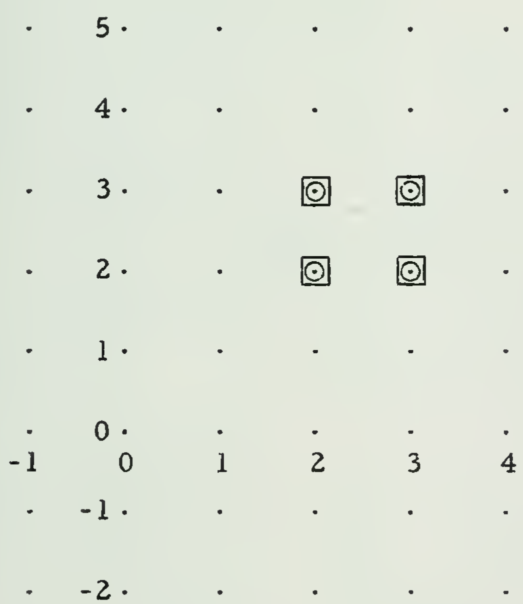
It so happens that if the intersection of two sets is empty, then the union of the two sets has a number of elements which is the <u>sum</u> of the number of elements in each set.



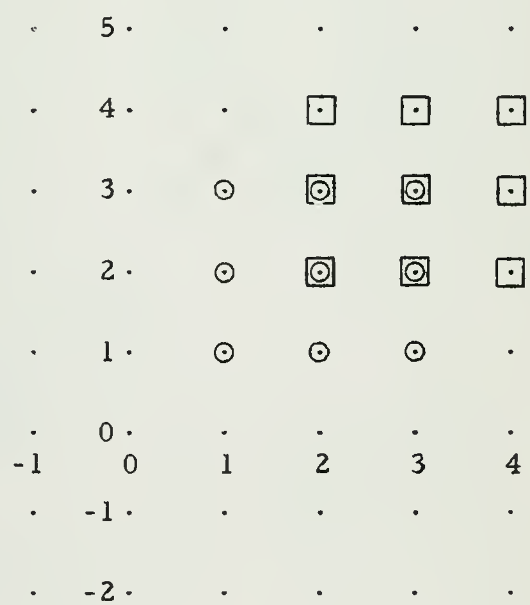
Set I (9 pts.)



Set II (9 pts.)



Intersection of
Set I and Set II (4 pts.)



Union of
Set I and Set II (14 pts.)

Thus, the solution sets for Exercise 2 are:

Set I: $\{(\square, \hexagon): -7 < \square < -2 \text{ and } -4 < \hexagon < 3\}$

Set II: $\{(\square, \hexagon): -8 < \square < 0 \text{ and } -4 < y < 0\}$

[We could have used letters rather than frames. For example, we could have written the description of Set I thus:

$$\{(x, y): -7 < x < -2 \text{ and } -4 < y < 3\}.]$$

Before the students begin Exercise 3 discuss with them the material given in T. C. 15D. Then ask that they examine each of the remaining exercises of Part C and write a description for the sets using this set notation. [The correct descriptions are under the sketch for each exercise in the Commentary for pages 4-16 and 4-17.]

> Be sure to point out that for these exercises we must adopt the special convention that the domain of the index is the set of ordered pairs of positive and negative whole numbers and 0. These numbers are also called integers.

pairs of directed numbers, and thus such sentences are sometimes called *set selectors*. And we can use the set selector itself in constructing a name (or: description) of its solution set. This is how it is done:

a) First, write a left-brace: $\{$. This indicates that one is going to write the name of a set.

b) Next, write an "ordered pair of pronumerals" [this symbol is called *an index*], and a colon immediately after the index. For example: $\{(x, y):$, or: $\{(a, b):$, or: $\{(p, q):$, or: $\{(\square, \hexagon):$, etc. Symbols such as these indicate that the set in question consists of ordered pairs. By convention, these ordered pairs are selected from the set of all ordered pairs of directed numbers. That is, the domain of '(x, y)' [or: '(a, b)', or: '(p, q)', or: '$(\square, \hexagon)$'] is understood [*by convention*] to be the set of all ordered pairs of directed numbers. The colon indicates that the writer is about to tell how the elements in the set are to be selected.

c) The next part of the name is the set selector itself, followed by a right-brace to indicate that the description is complete:

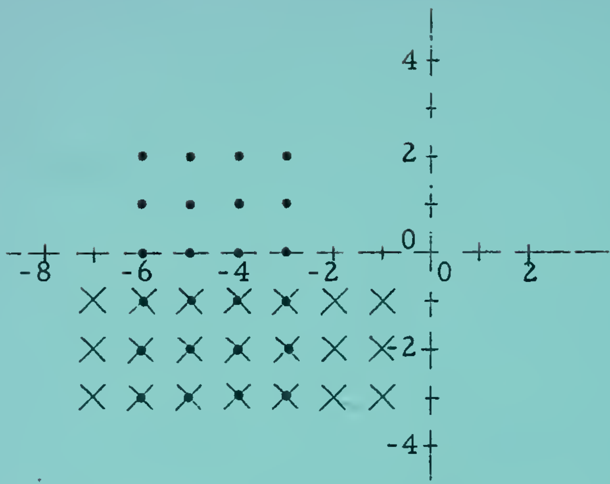
$$\{(x, y): \ -2 < x < 3 \ \text{and} \ 3 < y < 6\}$$

or: $$\{(\square, \hexagon): \ -2 < \square < 3 \ \text{and} \ 3 < \hexagon < 6\}.$$

[This symbol is read as 'the set of all ordered pairs [of directed numbers] (x, y) [or $(\square, \hexagon)$] such that $-2 < x < 3$ and $3 < y < 6$'. The symbol '$\{(x, y): \ldots\}$' is sometimes called a 'set-abstraction operator'.]

(continued on T. C. 15E)

2.



Set I: $\{(x, y): -7 < x < -2 \text{ and } -4 < y < 3\}$

Set II: $\{(x, y): -8 < x < 0 \text{ and } -4 < y < 0\}$

Intersection: 12

Union: 33

* * *

After students have struggled through the words of Exercises 1 and 2, they will welcome the pronumeral description of sets introduced after Exercise 2.

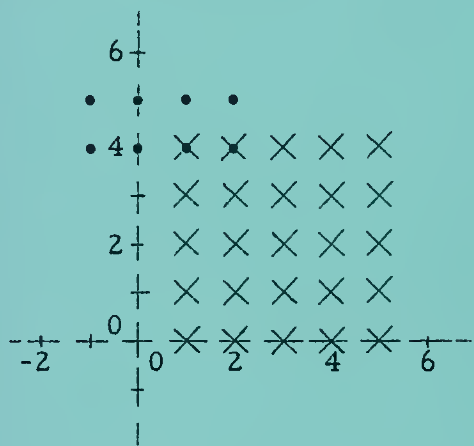
* * *

There is another way of shortening the description of these sets. As explained in Unit 3, a compound sentence such as '$-2 < x < 3$ and $3 < y < 6$' can be thought of as selecting from the set of all ordered pairs of directed numbers the set of those ordered pairs such that the first component is greater than -2 but less than 3 and the second component is greater than 3 but less than 6. [<u>Note well</u>: <u>In these</u> exercises of Part C, since we are working in the lattice plane, the ordered pairs which satisfy the above sentence must be just those whose components are whole numbers.] In fact each sentence can be thought of as selecting its solution set from the set of all ordered

(continued on T. C. 15D)

In the sketches which follow, we have used ' • 's to indicate the points of Set I, and ' X 's to indicate the points of Set II. The points in the intersection of Sets I and II will therefore be indicated by ' ⋇ 's. The number of points in the intersection and in the union are listed below the sketch.

1.



Set I: $\{(x, y): -2 < x < 3 \text{ and } 3 < y < 6\}$

Set II: $\{(x, y): 0 < x < 6 \text{ and } -1 < y < 5\}$

Intersection: 2

Union: 31

(continued on T. C. 15C)

Quick review quiz.

Write an equivalent expression which does not contain parentheses or other symbols of grouping.

1. $5(2a - 7b)$

2. $(x + 8y) \times (-3)$

3. $\frac{1}{3}(12r - 3s)$

4. $(-21m + 30x) \div 3$

5. $- \square \,(4 \triangle - 6 \hexagon)$

6. $(3.5u - 1.2v) \times (-6)$

7. $5ab(-3a - 11c)$

8. $(18e - 39f) \div (-3)$

9. $\frac{9m + 3n}{-6}$

10. $(-32r - 16s) \times \frac{1}{8}$

11. The expression '$F = \frac{9C}{5} + 32$' is used in converting from degrees C to degrees F. What is the temperature in degrees F corresponding to 22 degrees C?

12. $\{x: -3 < x < 3\}$ is a subset of which of these sets?

a) $\{x: -2 < x < 2\}$

b) $\{x: |x| < 4\}$

c) $\{x: -3 \geq x \geq 3\}$

d) $\{x: |x| \geq 3\}$

* * *

Students should be given plenty of time to do these exercises. They need freedom to exercise their intuition. There will be plenty of "hunting and trying". We know of no exercises which do a better job of training a student in reading symbols and in applying the basic meaning of graphing pairs of numbers. Exercise 15 is probably the best example of how to compel a student to ask himself the meanings of symbols.

* * *

(continued on T. C. 15B)

In each of the following exercises you are given descriptions of two sets. In each exercise plot the points in each set in such a way that you can tell them apart on a diagram of a plane lattice. (Remember that you have only points with whole number coordinates.)

(a) Tell the number of points in the intersection of the two given sets, and

(b) tell the number of points in the union of the two given sets.

1. Set I: All points with abscissa greater than -2 but less than 3 and ordinate greater than 3 but less than 6.

   Set II: All the points with abscissa greater than 0 but less than 6 and ordinate greater than -1 but less than 5.

2. Set I: All the points with abscissa greater than -7 but less than -2 and ordinate greater than -4 but less than 3.

   Set II: All the points with first coordinate greater than -8 but less than 0 and second coordinate greater than -4 but less than 0.

   Note: We can shorten our descriptions of sets of points by using pronumerals. For example, we can describe Set I of Exercise 2 as follows:

   The graph of every (□ , ⬡) such that

   $$-7 < \square < -2 \text{ and } -4 < ⬡ < 3.$$

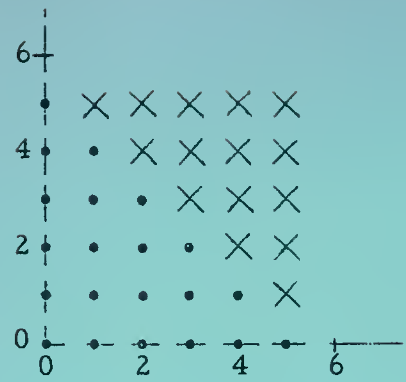
   Using letters, we can describe Set II of Exercise 2 in the same way:

   The graph of every (x, y) such that

   $$-8 < x < 0 \quad \text{and} \quad -4 < y < 0.$$

   Of course, for these exercises, we replace 'x' and 'y' by numerals for whole numbers only.

8.



Set I: $\{(x, y): \ x + y \leq 5 \ \text{and} \ x \geq 0 \ \text{and} \ y \geq 0\}$

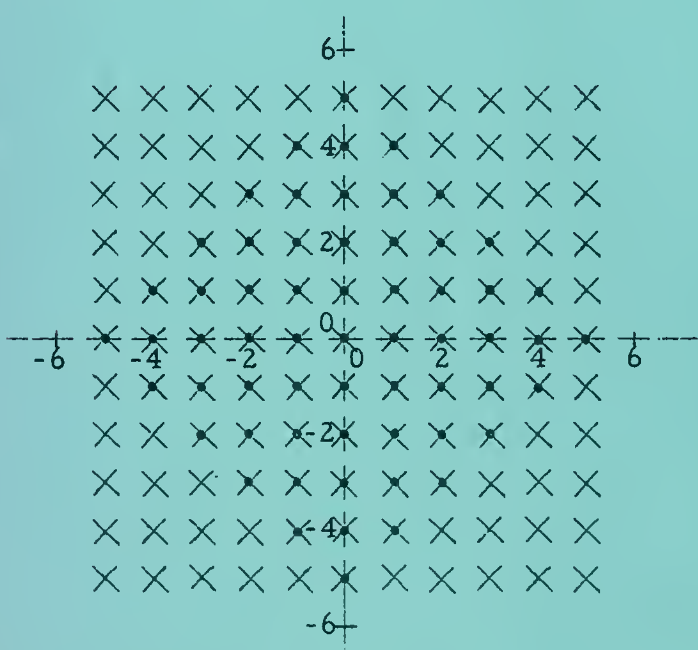
Set II: $\{(x, y): \ x + y > 5 \ \text{and} \ x < 6 \ \text{and} \ y < 6\}$

Intersection: 0

Union: 36

[Set I has 21 points and Set II has 15 points.]

9.



Set I: $\{(x, y): \ |x| + |y| \leq 5\}$

Set II: $\{(x, y): \ |x| \leq 5 \ \text{and} \ |y| \leq 5\}$

Intersection: 61

Union: 121

[Set I has 61 points and Set II has 121 points.]

5. Set I: $\{(x, y): -4 < x < -3 \text{ and } -2 < y < -1\}$
Set II: $\{(x, y): 3 < x < 4 \text{ and } 1 < y < 3\}$

Intersection: 0

Union: 0

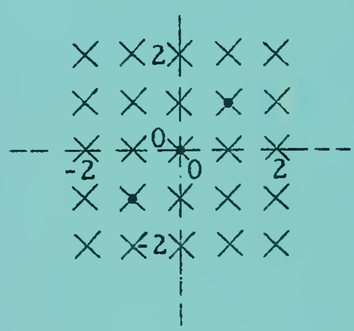
6. Set I: $\{(x, y): -1 < x < 0 \text{ and } 5 < y < 6\}$
Set II: $\{(x, y): 5 < x < 6 \text{ and } -1 < y < 0\}$

Intersection: 0

Union: 0

[In Exercises 5 and 6 sets I and II are both empty sets.]

7.



Set I: $\{(x, y): x = y \text{ and } |x| < 2 \text{ and } |y| < 2\}$
Set II: $\{(x, y): |x| < 3 \text{ and } |y| < 3\}$

Intersection: 3

Union: 25

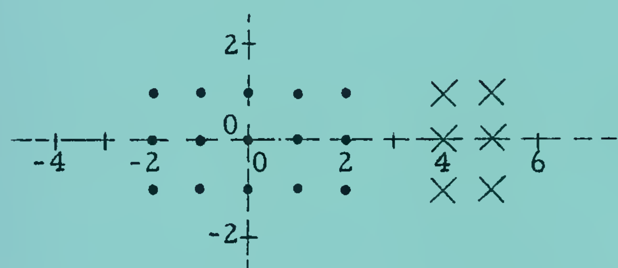
(continued on T. C. 16C)

To help in the consideration of the use of 'and' and 'or' in these exercises, have students consider the implications of these sentences:

1) Give me all the red balls or all the blue balls in your collection.
2) Give me all the balls that are red or blue in your collection.
3) Give me all the red balls and all the blue balls in your collection.

* * *

3.



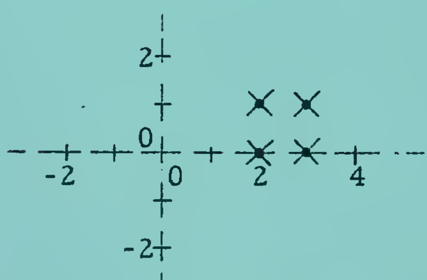
Set I: $\{(x, y): \ |x| < 3 \text{ and } |y| < 2\}$

Set II: $\{(x, y): \ 3 < x < 6 \text{ and } |y| < 2\}$

Intersection: 0

Union: 21

4.



Set I: $\{(2,0), (2,1), (3,0), (3,1)\}$

Set II: $\{(x, y): \ 1 < x < 4 \text{ and } -1 < y < 2\}$

Intersection: 4 [Set I = Set II]

Union: 4

(continued on T. C. 16B)

3. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$|x| < 3$ and $|y| < 2$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$3 < x < 6$ and $|y| < 2$.

4. <u>Set</u> <u>I</u>: The graph of (2, 0), (2, 1), (3, 0), and (3, 1).

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$1 < x < 4$ and $-1 < y < 2$.

5. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$-4 < x < -3$ and $-2 < y < -1$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$3 < x < 4$ and $1 < y < 3$.

6. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$-1 < x < 0$ and $5 < y < 6$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$5 < x < 6$ and $-1 < y < 0$.

7. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$x = y$ and $|x| < 2$ and $|y| < 2$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$|x| < 3$ and $|y| < 3$.

8. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$x + y \leq 5$ and $x \geq 0$ and $y \geq 0$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$x + y > 5$ and $x < 6$ and $y < 6$.

9. <u>Set</u> <u>I</u>: The graph of every (x, y) such that

$|x| + |y| \leq 5$.

<u>Set</u> <u>II</u>: The graph of every (x, y) such that

$|x| \leq 5$ and $|y| \leq 5$.

(continued on next page)

Here are supplementary exercises submitted by Mrs. Catlow's students:

a) Tell the number of points in the intersection of the two given sets, and

b) Tell the number of points in the union of the two given sets.

1) Set I: $\{(x, y): |x| + |y| = 7\}$
Set II: $\{(x, y): x > 1 > 0$ <u>and</u> $y > 6 > 5\}$

2) Set I: $\{(x, y): -|x| + y = 6\}$
Set II: $\{(x, y): 4 > x \geq 2$ <u>or</u> $4 < y < 9\}$

* * *

<u>Quiz.</u>

Simplify.

1. $3(m - 3n) + 2(4n - m)$
2. $-8(a + 5c) - (6a - 7c)$
3. $(4\square - 5\triangle) \times 6 - 7(\triangle + 3\square)$
4. $\frac{1}{3}(15x + 12y) - \frac{1}{5}(5x - 15y)$
5. $\frac{-9e + 27f}{3} + \frac{-9e + 36f}{9}$
6. $(14.7r - 29.4s) \div 7 + (3.2s - 5.6r) \div (-8)$
7. Which of the following equations has more than one root?

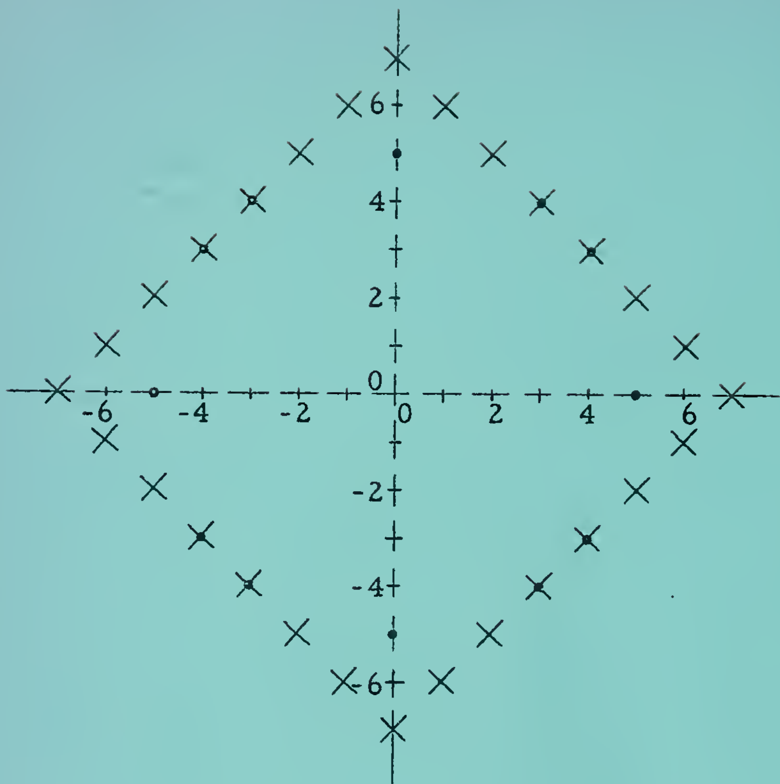
   a) $|13 - 8x| + 32 = 7$
   b) $|9 + m| = 0$
   c) $|4r - 3| + 7 = 2 + |3 - 4r|$
   d) $|5 + 3x| + 9 = 45$
   e) $|a + 5.2| + 7.2 = 3$

8. The volume of a cube is $\frac{8}{27}$ cubic feet. What is the length of one of its edges?

9. A large can of pineapple juice has a diameter of 4 inches and is 8 inches high. How many cubic centimeters of pineapple juice can the can hold? [1 inch $\stackrel{a}{=}$ 2.54 centimeters.]

10. A contractor employed 150 men with a daily payroll of $2870. The unskilled laborers earned $15 per day and the skilled laborers $25 per day. How many of each did he employ?

15.



Set I: $\{(x, y): xx + yy = 25\}$

Set II: $\{(x, y): |x| + |y| = 7\}$

Intersection: 8

Union: 32

16. Set I: $\{(x, y): 2y + 2x = 1\}$

Set II: $\{(x, y): 3y - 3x = 1\}$

Intersection: 0

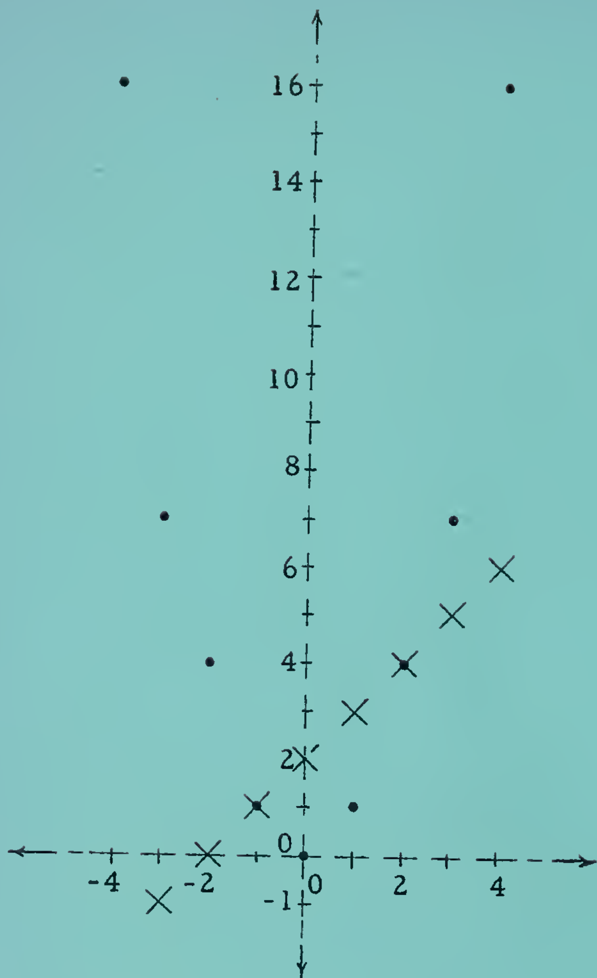
Union: 0

[Both Set I and Set II are empty sets.]

* * *

(continued on T. C. 17E)

14.



Set I: $\{(r, s): \ s = rr\}$

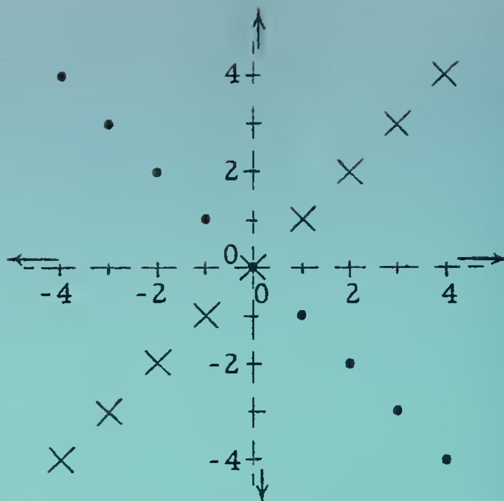
Set II: $\{(p, q): \ q = p + 2\}$

Intersection: 2

Union: indefinitely many

(continued on T. C. 17D)

11.



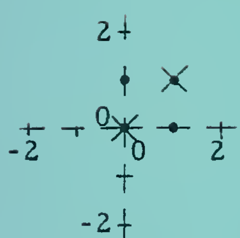
Set I: $\{(x, y): \ x + y = 0\}$

Set II: $\{(x, y): \ x - y = 0\}$

Intersection: 1

Union: indefinitely many

12.



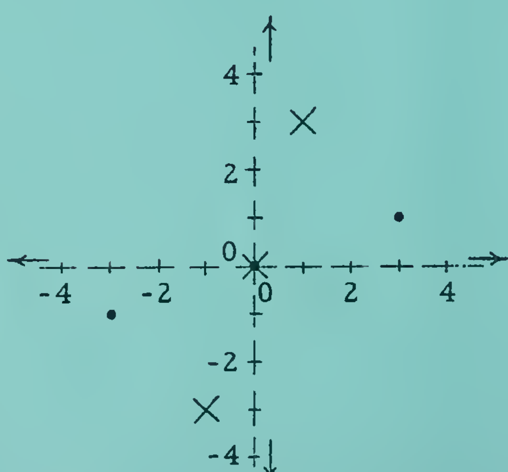
Set I: $\{(x, y): \ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

Set II: $\{(x, y): \ y = x \text{ and } y = xx \text{ and } y = xxx \text{ and } y = xxxx\}$

Intersection: 2

Union: 4

13.



Set I: $\{(a, b): \ a = 3b\}$

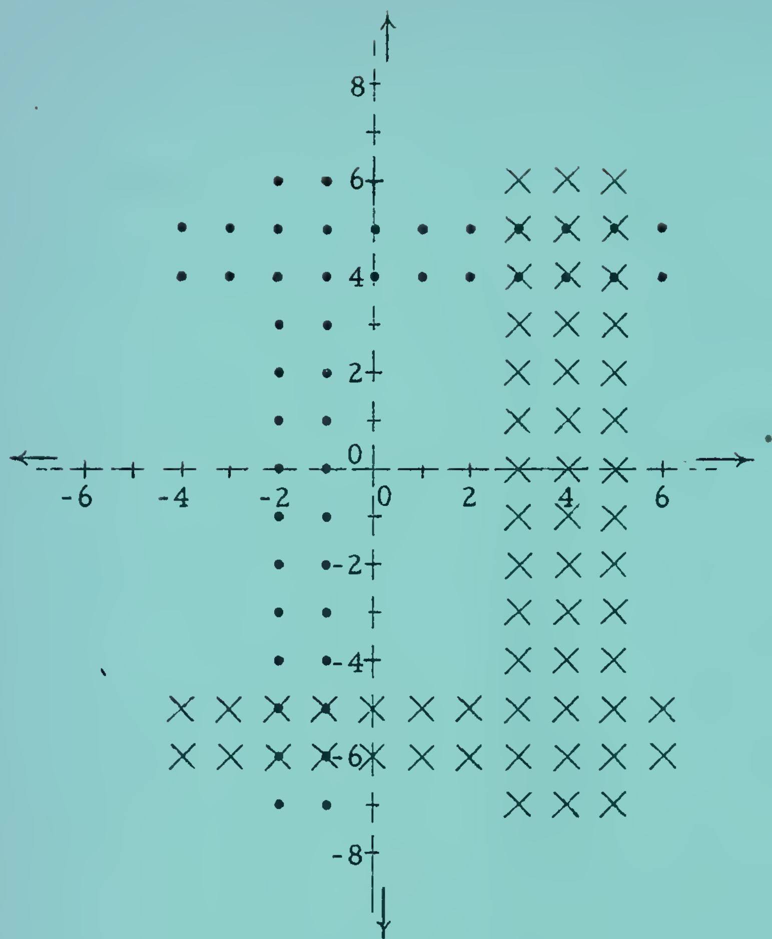
Set II: $\{(a, b): \ b = 3a\}$

Intersection: 1

Union: indefinitely many

(continued on T. C. 17C)

10.



Set I: $\{(x, y): \ -3 < x < 0 \ \text{ or } \ 3 < y < 6\}$

Set II: $\{(x, y): \ 2 < x < 6 \ \text{ or } \ -4 > y > -7\}$

Intersection: 10

Union: indefinitely many

[Note the use of the word 'or' in these descriptions and compare its use with that of the word 'and' in Exercise 8, for example. The word 'or' gives you a union; 'and' gives you an intersection.]

(continued on T. C. 17B)

10. Set I: The graph of every (x, y) such that
$-3 < x < 0$ or $3 < y < 6$.

Set II: The graph of every (x, y) such that
$2 < x < 6$ or $-4 > y > -7$.

11. Set I: The graph of every (x, y) such that
$x + y = 0$.

Set II: The graph of every (x, y) such that
$x - y = 0$.

12. Set I: The graph of every (x, y) such that
$0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Set II: The graph of every (x, y) such that
$y = x$ and $y = xx$ and $y = xxx$ and $y = xxxx$.

13. Set I: The graph of every (a, b) such that
$a = 3b$.

Set II: The graph of every (a, b) such that
$b = 3a$.

14. Set I: The graph of every (r, s) such that
$s = rr$.

Set II: The graph of every (p, q) such that
$q = p + 2$.

15. Set I: The graph of every (x, y) such that
$xx + yy = 25$.

Set II: The graph of every (x, y) such that
$|x| + |y| = 7$.

16. Set I: The graph of every (x, y) such that
$2y + 2x = 1$.

Set II: The graph of every (x, y) such that
$3y - 3x = 1$.

Part D is another set of exercises designed to familiarize the student with ordered pairs and graphs. Techniques become habituated as the student immerses himself in the "game"--a pleasant way of teaching.

* * *

You should illustrate Sample 1 on a lattice diagram indicating with dotted line segments the path moved each time. Ask the students to <u>predict</u> the final position at the end of the tenth move or the twentieth move.

* * *

In most of these exercises the student is concerned with what happens to a single point or to a few points. However, one could ask, as in Exercises 13 and 14, about what happens to <u>every</u> point in the lattice. Some "moves" could be interpreted as a sliding or turning of the entire plane. [Think of a piece of transparent lattice paper sliding above another.] With this interpretation, our moves are transformations of the plane into itself.

* * *

Mr. Marston reported that when his class did the game described in Sample 2, someone raised the question as to whether there was a "pattern". After some discussion of this, one student realized that if the picture were folded on its "x = y-line", one set would fall on top of the other set.

* * *

The students can keep track of moves by drawing dotted line segments between consecutive positions. Eventually the student will tire of the physical act of plotting points and will arrive at final positions by strictly arithmetic procedures. Do not rush them into this latter technique.

D. "Plane Lattice Games"

A plane lattice game consists of a series of "jumps" or moves from point to point of the lattice with each move made according to a given rule.

Sample 1. Rule: For every (x, y), we define 'a jump' to be a move from the graph of (x, y) to the graph of (x + 1, y + 1).

Start at the graph of (-3, -4) and make 5 jumps.

Solution. First jump: From the graph of (-3, -4) to the graph of (-3 + 1, -4 + 1) or (-2, -3).

Second jump: From the graph of (-2, -3) to the graph of (-2 + 1, -3 + 1) or (-1, -2).

Third jump: From the graph of (-1, -2) to the graph of (-1 + 1, -2 + 1) or (0, -1).

Fourth jump: From the graph of (0, -1) to the graph of (1, 0).

Fifth jump: From the graph of (1, 0) to the graph of (2, 1).

Sample 2. Rule: For every (x, y), a jump is a move from the graph of (x, y) to the graph of (y, x).

Consider the set whose elements are the graphs of:

(2, 1), (4, 0), (6, -1).

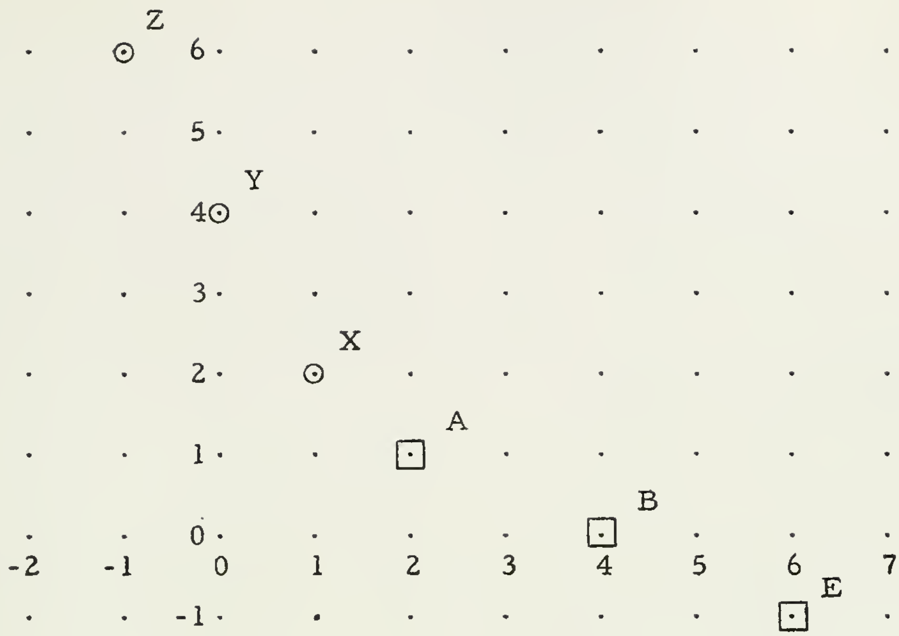
Move each of these points one jump and describe the new set of points.

Solution. A: (2, 1) jumps to X: (1, 2)

B: (4, 0) jumps to Y: (0, 4)

C: (6, -1) jumps to Z: (-1, 6)

(continued on next page)



The new set consists of the points which are the graphs of:

(1, 2), (0, 4), (-1, 6).

Sample 3. Rule: For every (x, y), a jump is a move from the graph of (x, y) to the graph of (y, x).

Consider the points in the set:

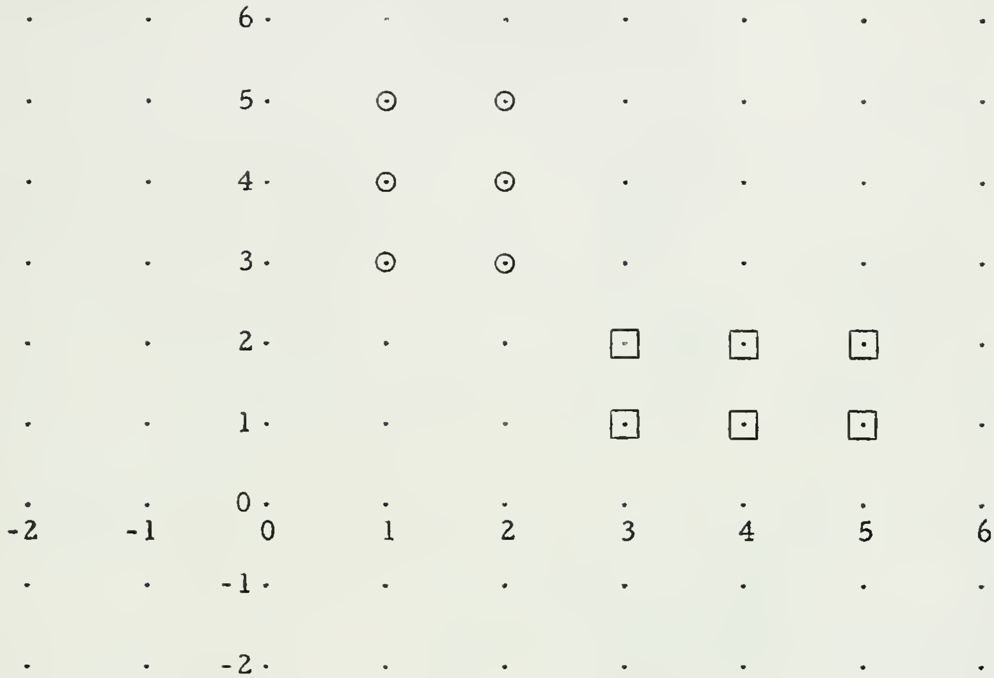
The graph of every (a, b) such that

$2 < a < 6$ and $0 < b < 3$.

Let each point in the set make one jump and describe the new set of points.

Solution. The points in the given set are "boxed" and the points in the new set are "circled". (See diagram on next page.)

(continued on next page)



The point for (3, 2) in the old set jumps to
the point for (2, 3) in the new set;
the point for (3, 1) in the old set jumps to
the point for (1, 3) in the new set; etc.

We can describe the new set of points as follows:

The graph of every (a, b) such that

$0 < a < 3$ and $2 < b < 6$.

1. Rule: For every (x, y), a move is a jump from the graph of (x, y) to the graph of (x, y - 2).

   Start at the graph of (0, 4) and make 3 moves. Give the coordinates of the final position.

2. Rule: For every (x, y), a move is a jump from the graph of (x, y) to the graph of (x + 2, y - 3).

   Start at the graph of (3, 3) and make 3 moves. Give the coordinates of the final position.

(continued on next page)

Be sure that the students understand the abbreviation by having them give, occasionally throughout the exercises, the longer statement of the rule.

* * *

Exercises 6 and 7 give a little "extra" review in solving equations. You can make more exercises like these without very much effort. In constructing such ~~problems~~ be sure you observe the restriction to integer coordinates, although an occasional "impossible" is good for the mental health and academic adjustment of the student!

* * *

Here are two other games suggested by Mrs. Catlow's students. They imagined a grasshopper making these "jumps".

1) $(x, y) \rightarrow (2x - 2, 4 - 2y)$

   After making 4 jumps, the grasshopper landed on $(34, -100)$. Where did he start?

2) $(u, v) \rightarrow (2u + 5, 3 - 4v)$

   After making 3 jumps, the grasshopper landed on $(43, 231)$. Where did he start?

<u>Note</u>: In the following exercises we shall give an abbreviated form for each rule. For example, the statement of the rule in Exercise 2 could have been abbreviated as:

$$(x, y) \rightarrow (x + 2, y - 3).$$

3. Rule: $(x, y) \rightarrow (2x, 2y)$.
   Start at the graph of $(1, 2)$, make 5 moves, and give the coordinates of the final position.

4. Rule: $(x, y) \rightarrow (3x, 2y)$.
   Start at the origin, make 10 moves, and give the coordinates of the final position.

5. Rule: $(j, k) \rightarrow (3j - 5, 2k + 3)$.
   Start at the graph of $(4, -3)$, make 4 moves, and give the coordinates of the final position.

6. Rule: $(x, y) \rightarrow (x - 3, y + 2)$.
   After making 3 moves, the final position is the graph of $(-4, 1)$. Give the coordinates of the starting point.

7. Rule: $(u, v) \rightarrow (2u + 5, 3 - 4v)$.
   After making 3 moves, the final position is the graph of $(43, -89)$. Give the coordinates of the starting point.

8. Rule: $(x, y) \rightarrow (x + y, x - y)$.
   Start at the graph of $(3, -3)$, make 4 moves, and give the coordinates of the final position.

(continued on next page)

and 1 is an integer and there is no largest integer. Hence, the derived set is equal to the given set, and we could use the same name for the derived set as we did for the given set. However, it is instructive to examine a procedure for obtaining a name for the derived set in a more mechanical fashion. Here is such a name:

$$\{(m + 1,\ n + 1):\ m \text{ and } n \text{ are integers and } m + n = n + m\}.$$

Note the "index": $(m + 1,\ n + 1)$. Suppose the ordered pair (8, 5) is up for selection or rejection by the set selector. Instead of judging (8, 5), the selector actually judges (7, 4). Since it is customary to use an expression such as '(m, n)' as an index instead of '(m + 1, n + 1)', the conventional name for the derived set is:

$$\{(m, n):\ m - 1 \text{ and } n - 1 \text{ are integers and } (m - 1) + (n - 1) = (n - 1) + (m - 1)\}.$$

Clearly, the set selector here is equivalent to the set selector in the name of the given set. This procedure for constructing names for derived sets is more useful in cases in which the given set is finite.

13. The given set is $\{(y,\ z):\ y \text{ and } z \text{ are integers and } yz = zy\}$. A name for the derived set is obtained as follows:

$$\{(y + \tfrac{1}{2},\ z + \tfrac{1}{2}):\ y \text{ and } z \text{ are integers and } yz = zy\}$$
$$= \{(y,\ z):\ y - \tfrac{1}{2} \text{ and } z - \tfrac{1}{2} \text{ are integers and } (y - \tfrac{1}{2})(z - \tfrac{1}{2}) = (z - \tfrac{1}{2})(y - \tfrac{1}{2})\}.$$

Since there is no point with integral components which satisfies the set selector, the derived set is the empty set.

Exercises 9 through 14 provide excellent practice in using the set abstraction operator notation. In each exercise, you are given a set and a transformation rule. After the student has handled the problems geometrically, he should try to construct names for the given sets and the derived sets. [Use the convention that the domain of the index is the set of all ordered pairs of directed numbers. Thus, the plane lattice itself is $\{(x, y):$ x and y are integers$\}$.]

9. Given set:

$\{(a, b):$ a and b are integers and $a = b$ and $-5 \leq a \leq 0$ and $-5 \leq b \leq 0\}$

Derived set:

$\{(a, b):$ a and b are integers and $a = b$ and $0 \leq a \leq 5$ and $0 \leq b \leq 5\}$

10. $\{(x, y):$ x and y are integers and $x = 2$ and $-3 \leq y \leq 0\}$
$\{(x, y):$ y and x are integers and $y = 2$ and $-3 \leq x \leq 0\}$

11. (a) $\{(x, y):$ x and y are integers and $[(x = 4$ and $y = 3)$ or $(x = 5$ and $y = -1)$ or $(x = 6$ and $y = -2)]\}$

The derived set is equal to the given set.

(b) The derived set is $\{(x, y):$ y and x are integers and $[(y = 4$ and $x = 3)$ or $(y = 5$ and $x = -1)$ or $(y = 6$ and $x = -2)]\}$.

(c) The derived set is equal to the given set.

(d) As in (c).

(e) As in (b).

12. The given set is $\{(m, n):$ m and n are integers and $m + n = n + m\}$. This set is the plane lattice itself. Each point in this set is jumped to another point in the set because the sum of an integer

(continued on T. C. 22B)

9. Rule: $(s, t) \rightarrow (|s|, |t|)$.

Take for one move each point in the set:

The graph of every (a, b) such that

$$a = b \text{ and } -5 \leq a \leq 0 \text{ and } -5 \leq b \leq 0$$

and describe the new set of points.

10. Rule: $(x, y) \rightarrow (y, x)$.

Move one time each point in the set of graphs of:

$$(2, 0), (2, -1), (2, -2), (2, -3)$$

and describe the new set of points.

11. Rule: $(x, y) \rightarrow (y, x)$.

(a) Move <u>twice</u> each point in the set of graphs of:

$$(4, 3), (5, -1), (6, -2)$$

and describe the new set of points.

(b) Move three times each point in the set given in (a) and describe the new set of points.

(c) Repeat (b) but move each point four times.

(d) Repeat (b) but move each point an even number of times.

(e) Repeat (b) but move each point an odd number of times.

12. Rule: $(y, x) \rightarrow (y + 1, x + 1)$

Move once each point in the set:

The graph of every (m, n) such that

$$m + n = n + m$$

and describe the new set of points.

13. Rule: $(x, y) \rightarrow (x + \frac{1}{2}, y + \frac{1}{2})$.

Move once each point in the set:

The graph of every (y, z) such that

$$yz = zy$$

and describe the new set of points.

(continued on next page)

Quiz.

Find the elements in each set.

1. $\{\square : 11\square + 3 = 135\}$

2. $\{w: w - 10w = 96\}$

3. $\{r: \frac{1}{2}r - 13 = 65\}$

4. $\{m: \frac{m}{12} + 18 = 126\}$

5. $\{\Delta: -\frac{1}{3}\Delta + 10 = -\frac{3\Delta}{4} - 8 + \frac{5\Delta}{12}\}$

6. $\{h: 2.5h + 45 = \frac{5h + 92}{2} - 1\}$

7. $\{d: \frac{4}{3d - 8} = \frac{2}{5d + 4}\}$

8. $\{z: \frac{3z - 5}{z} = 0\}$

9. Dave is now two years older than his brother Joe. In eight years, the ratio of their ages will be 5:4. What is the present age of each?

10. Gene can complete a certain job in 5 days. When Dave works with him, the job can be done in 3 days. How long would it take Dave to do the job if he worked alone?

14. The g[illegible] s[illegible] common [illegible] circle [illegible] common [illegible] (0, 0) and [illegible] "translate" [illegible]

[illegible]

15. [illegible] ity of [illegible]
the plane lattice, or a [illegible] set
x + y = [illegible] y [illegible], will show [illegible]
set. No matter how complicated the [illegible]
points to [illegible] to the derived
set is the empty [illegible].

* * *

I [illegible] tion 1.03, [illegible] mention [illegible]
COURSE [illegible] small [illegible] the complete coordinat[illegible]

[illegible]

14. The given set contains twelve points arranged on the circle with center at (0, 0) and radius 5. The two moves "translate" the points in the given set to a circle with radius 5 and center at (4, -6).

given set:

$\{(c, d):$ c and d are integers and $cc + dd = 25\}$

first derived set:

$\{(c + 2,\ d - 3):$ c and d are integers and $cc + dd = 25\}$

second derived set:

$\{(c + 2 + 2,\ d - 3 - 3):$ c and d are integers and $cc + dd = 25\}$

The more customary name for the second derived set is:

$\{(c, d):$ $c - 4$ and $d + 6$ are integers and

$(c - 4)(c - 4) + (d + 6)(d + 6) = 25\}$.

15. The complexity of the rule is just a trick. A careful search of the plane lattice, or a critical examination of the set selector: $x = y + 1$ and $x = y + 2$, will show that the given set is the empty set. No matter how complicated the rule is, if there are no points to jump from the given set to the derived set, the derived set is the empty set.

* * *

In beginning section 4.03, mention to the students that in SECOND COURSE we shall call the complete coordinate plane:

the number plane.

Use 'the number plane' interchangeably with 'the coordinate plane'.

(continued on T. C. 23B)

14. Rule: $(x, y) \to (x + 2, y - 3)$.

Move twice each point in the set:

The graph of every (c, d) such that

$$cc + dd = 25$$

and describe the new set of points.

15. Rule: $(x, y) \to (3xx - 7y, 3yy - 7x)$.

Move once each point in the set:

The graph of every (x, y) such that

$$x = y + 1 \quad \text{and} \quad x = y + 2$$

and describe the new set of points.

4.03 The complete coordinate plane.--Up to this point you have been working with plane lattices and with points which have whole number coordinates. It is natural to ask if it is possible to work with points which have coordinates belonging to the entire set of directed numbers. For example, can you plot the graph of $(3\frac{1}{2}, -16)$ or of $(-2.5, +7.5\%)$ or of $(+97, -\sqrt{92})$? The answer to this question is 'yes'. However, we do not use plane lattices for this purpose. Instead, we start with a plane lattice and imagine that the spaces between the points of the lattice are completely filled with other points. It is reasonable to believe that for any given ordered pair of directed numbers, you can locate a point which is the graph of the ordered pair. Similarly, given any point in a "completely filled" plane lattice, you can find a pair of directed numbers to serve as coordinates of this point.

A plane lattice which has been filled completely is called a Cartesian* coordinate plane or, simply, a coordinate plane. As in the case of a plane lattice with indefinitely many points, it is impossible to draw an accurate diagram of a coordinate plane. Even if you draw a diagram

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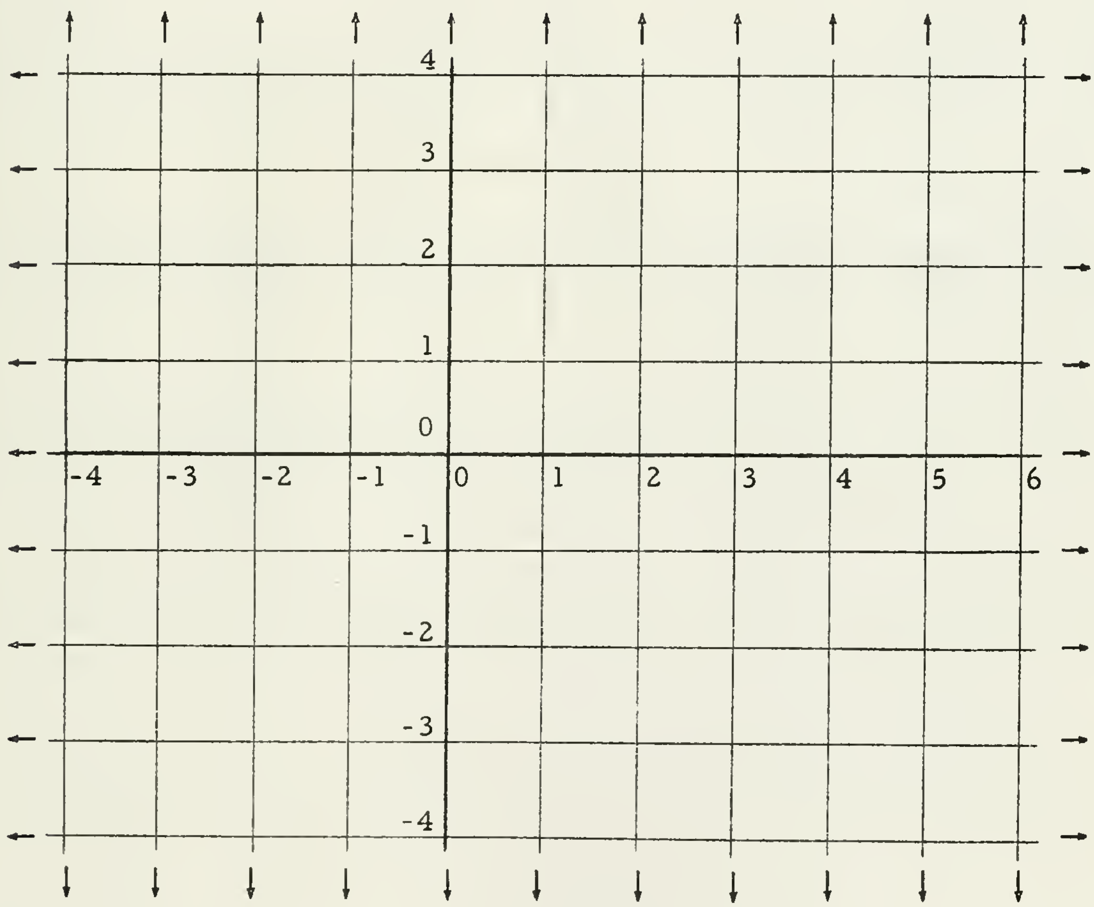
*'Cartesian' is derived from the Latinized form of the name of Rene Descartes (dā-kärt´), a French mathematician and philosopher who lived in the first half of the seventeenth century.

of only part of a coordinate plane, you could not show all the points in that part of the plane because all you would have on your paper is a completely black region. So, when we make a diagram of part of a coordinate plane, we show only some of the points in that part. We show just a few sets of points such that each set consists of points having either the same first coordinate or the same second coordinate. Such sets of points are shown as straight lines in the diagram; these straight lines are sometimes called grid lines.



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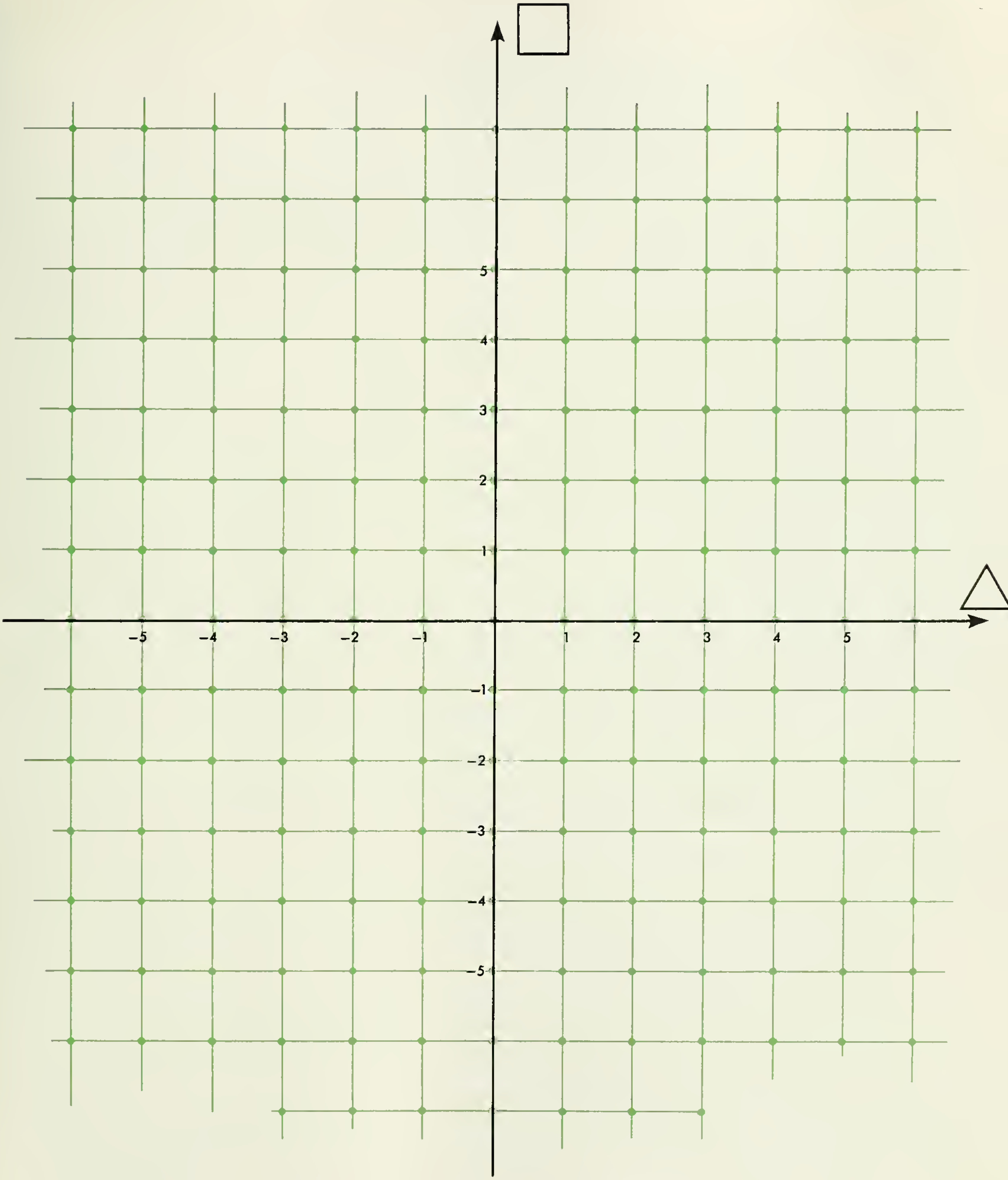


COORDINATE PLANE PAPER A

NAME ______________________________

COORDINATE PLANE PAPER B

Note in the diagram on the preceding page that the coordinate axes are made more prominent in appearance than the other grid lines so that they can be located more easily. Study the diagram and answer these questions.

(1) Find the grid line which contains points with first coordinate 3.

(2) Find the grid line which contains points with second coordinate -2.

(3) Point to the grid line which contains points with ordinate 4.

(4) Point to the grid line which contains points with abscissa -1.

(5) Point to the grid line which contains points with second coordinate 0.

(6) Draw the grid line which contains points with first coordinate $\frac{1}{2}$.

(7) Draw the grid line which contains points with second coordinate $-2\frac{1}{3}$.

(8) Point to the grid lines which contains the graph of (5, 3). Of (2, -1). Of (-3, -4). Of (4, 4). Of $(\frac{1}{2}, 2)$.

(9) Draw the grid lines which contain the graph of $(3\frac{1}{2}, 2\frac{1}{2})$. Of (-2.5, 1.5). Of (-3.5, -3.5).

In making a diagram of part of a coordinate plane you are completely free in your choice of which grid lines to draw. (It is customary, however, to include the coordinate axes.) Usually, you make a diagram for a particular problem and the coordinates of the points to be plotted for the problem determine the selection of grid lines. For example, the diagram on page 4-24 would be completely useless if you wanted to plot the graph of say, (10, 17). When you use cross section paper (or <u>graph paper</u>, as it is commonly called), you will find equally spaced grid lines already printed on the paper. After selecting two of these grid lines as coordinate axes, you can then decide upon the first or second coordinate to be assigned to each grid line. You do this by selecting a location for the graphs of (1, 0) and (0, 1). That is, you select

In an ordinary drawing of the number plane, paper-distance is proportional to number-plane-distance [to the limit of accuracy in your drawing]. In a drawing which assigns "different scales" to the two axes, horizontal paper-distance is proportional to number-plane-distance and vertical paper-distance is proportional to number-plane-distance, but the factors of proportionality are different. In certain cases, the two kinds of distance are not even proportional, e.g. when using logarithmic graph paper.

* * *

UICSM Coordinate Plane Paper A and B is ideally suited to the discussion of the difference between paper-distance and number-plane-distance. Suppose a student plots the point (3, 2) and the point (7, 6) on Coordinate Plane Paper A, and also plots these points on Coordinate Plane Paper B. The number-plane-distance between these two points is the same, namely 5, on both diagrams, but the paper-distances are different.

The problem of selecting an appropriate scale for making a graph is an eminently practical one. The student who goes on to do any kind of laboratory measurement work will find himself repeatedly examining the range of his measurements in order to choose a scale which will display the measurements appropriately. However, there are some possible misconceptions that may arise from the work on page 4-26. The number plane consists of points which correspond to all possible ordered pairs of real numbers. Distance in the number plane is defined (later) in terms of these pairs of numbers. Thus, the distance between (2, 1) and (6, 4) is 5 regardless of the picture you draw of the number plane. It is still 5 even if on a drawing it appears to be two feet, or if it appears to be one thirty-second of an inch. Thus, you must be careful to distinguish between two kinds of distance in these problems: The distance in the number plane which is computed from the components of points, and the distance which you measure with a ruler in the <u>picture</u> of the number plane with which you are working. When you pick different scales for the two axes in a drawing of the number plane, you are purposely making a distorted picture of the number plane to make it easier to work with the particular problem at hand.

It would be a good class exercise to discuss these two notions of distance and then decide what is meant by 'distance' in the last sentence at the top of page 4-26. You might call one kind of distance: 'paper-distance' and call the other kind of distance: 'number-plane-distance'. [The student should already be familiar from geography class with the distinction in meaning between 'ground distance' and 'map distance'. This is a related idea.]

(continued on T. C. 26B)

a <u>scale</u> for each axis. Here are several examples. Note that the scales differ from diagram to diagram even though the smallest distance between parallel grid lines is the same for all of the diagrams.

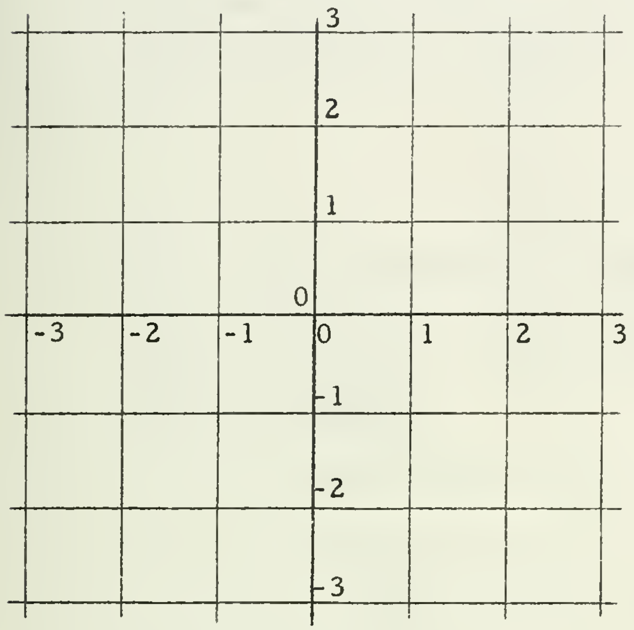


Diagram I

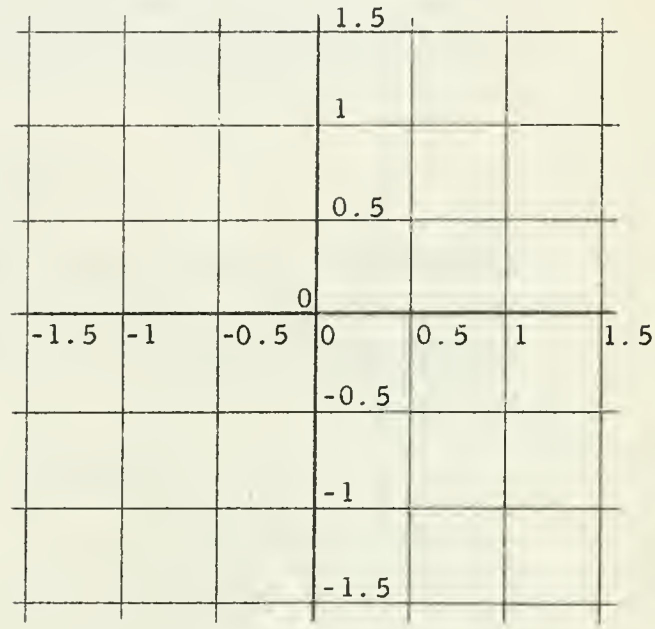


Diagram II

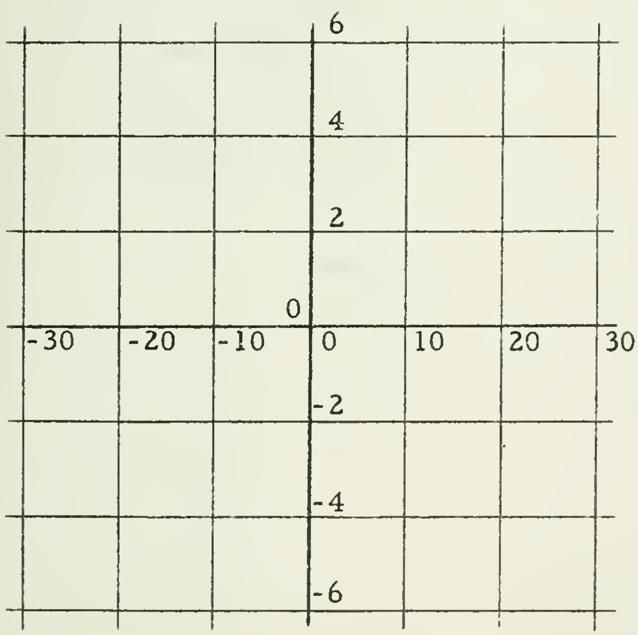


Diagram III

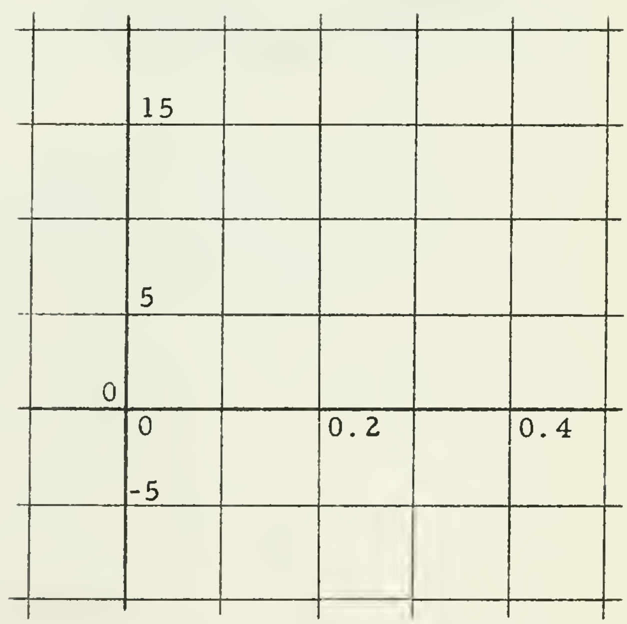


Diagram IV

In Diagram I the same scale is used on both axes. The same scale is used on both axes in Diagram II. The scales for the axes differ in Diagram III and in Diagram IV. Note (Diagram IV) that it is not necessary to label each grid line. When the grid lines are equally spaced, you can always tell the coordinates to be assigned to "in-between" grid lines.

EXERCISES

A. For each exercise draw a diagram of part of the coordinate plane and plot the graphs of the ordered pairs given. Select scales so that all the graphs in the exercise can be conveniently plotted on the same diagram.

1. (3, 5), (2, -1), (4, -3), (0, -2), (-3, 4)
2. (2.5, 3.5), (-1.5, 1.5), (0, -2.5), (-3.5, 1.5), (3, -2)
3. (20, 3), (-40, 5), (30, -4), (-50, 1), (0, -2)
4. (1, 1), (-1, -1), (2, 8), (-3, -27), (4, 64)
5. (0.3, 7), (-0.2, -3), (-1, 3), (-0.8, 0), (0.5, -5)

B. You can single out a polygon in the coordinate plane by giving the coordinates of its vertices. The line segments between these points are the sides of the polygon.

Sample. Plot the graphs of:

A: (1, 1); B: (3.5, 3); C: (5, 1)

and find the area and the perimeter of $\triangle ABC$.

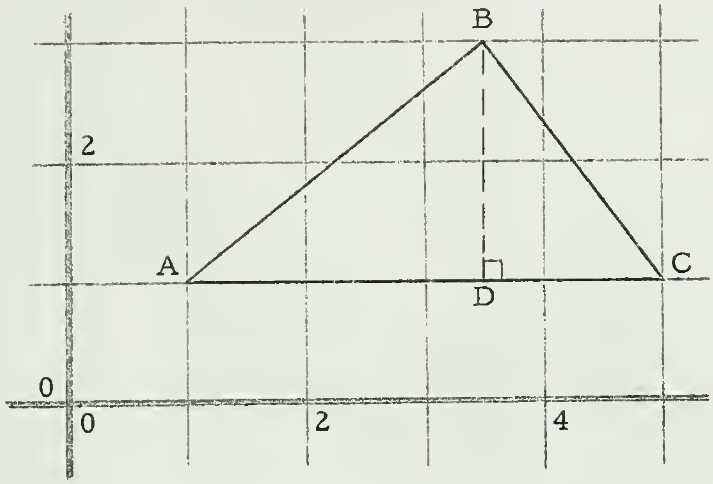
(continued on next page)

Review (or teach) whatever geometric concepts are necessary for this set of exercises. Students will need to know how to find approximations to square roots and how to apply the Pythagorean rule.

* * *

It is a good idea to review congruent triangles with the class by having them note that the area of ΔABC is actually the area of four little squares arranged along the segment AC. This kind of informal geometric discussion is beneficial and more readily handled in the classroom than in the textbook. You should engage in it whenever the opportunity arises.

Solution.



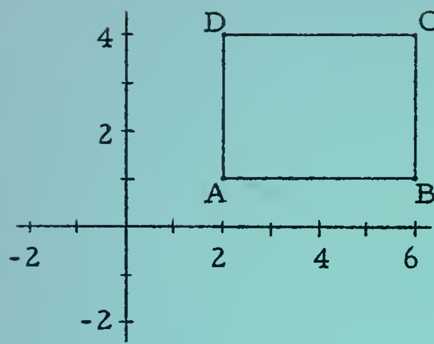
To find the area of $\triangle ABC$ we note that the base AC is 4 units long and that the height (the length of the line segment BD which is perpendicular to AC) is 2 units. Thus, the area is $\frac{1}{2}(2)(4)$ or 4 square units.

To find the perimeter of $\triangle ABC$, we first find the lengths of the three sides. The length of AC is 4 units. The length of AB can be found by using the Pythagorean rule since AB is the hypotenuse of right triangle ABD. Since the length of AD is 2.5 units and the length of BD is 2 units, then the length of AB in units is

$$\begin{aligned} &\sqrt{(2)(2) + (2.5)(2.5)} \\ &= \sqrt{4 + 6.25} \\ &= \sqrt{10.25} \\ &\overset{a}{=} 3.2 \end{aligned}$$

(continued on next page)

1.

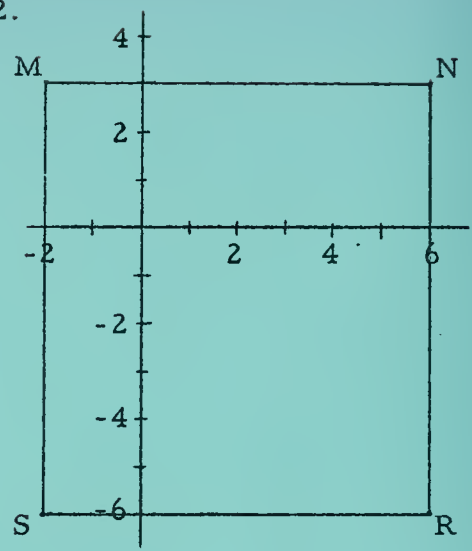


a) Rectangle

b) Perimeter: 14

c) Area: 12

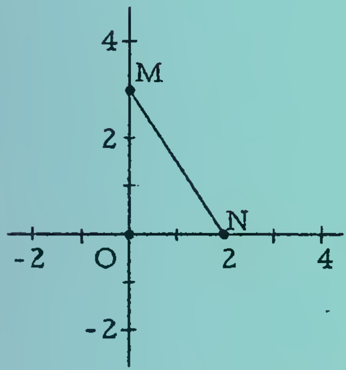
2.



a) Perimeter: 34

b) Area: 72

3.



a) Area: 3

b) Perimeter: $5 + \sqrt{13}$

Similarly, the length in units of BC is

$$\sqrt{(2)(2) + (1.5)(1.5)}$$
$$= \sqrt{4 + 2.25}$$
$$= \sqrt{6.25}$$
$$= 2.5$$

Hence, the perimeter is approximately
$4 + 3.2 + 2.5$ or 9.7 units.

Note: All of the lengths in the discussion above are measured in whatever units are used in making the diagram of the coordinate plane.

1. Plot the graphs of:

   A: (2, 1); B: (6, 1); C: (6, 4); D: (2, 4).

   Draw the line segments AB, BC, CD, and DA.

   (a) What is the figure ABCD called?

   (b) What is its perimeter?

   (c) What is its area?

2. Plot the graphs of:

   M: (-2, 3); N: (6, 3); R: (6, -6); S: (-2, -6).

   Draw the line segments MN, NR, RS, and SM.

   (a) What is the perimeter of the figure MNRS?

   (b) What is its area?

3. Plot the graphs of:

   M: (0, 3); N: (2, 0); O: (0, 0).

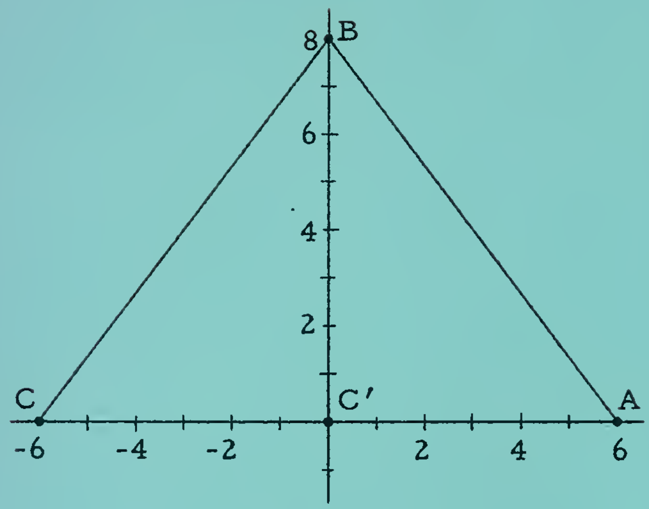
   Draw the line segment MN.

   (a) What is the area of $\triangle$MON?

   (b) What is the perimeter of $\triangle$MON?

(continued on next page)

5.



a) Perimeter: 32

b) Area: 48

6. a) 3 b) 4

c) 5 d) 5

e) 5 f) 5

g) 13 h) 13

i) 5 j) 25

k) $\sqrt{82}$ l) $\sqrt{197}$

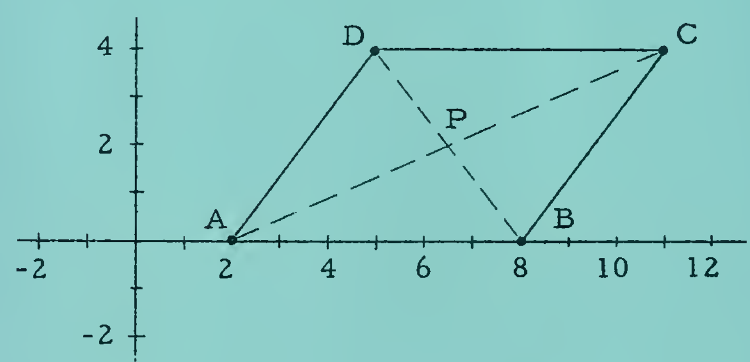
If your students had work in the eighth grade on similar triangles and congruent triangles, and if they recall this work, they should make interesting headway in part (e) of Exercise 4. It is quite within the student's ability to derive the mid-point formula for line segments from his knowledge of similar triangles.

* * *

In Exercise 6, we hope that students will be able to give the length of a line segment by using the coordinates of the end-points without resorting to graphing before they reach the end of this exercise.

* * *

4.



a) Parallelogram

b) $\overline{AD}$ is 5

c) Perimeter: 22

d) $\overline{AC}$ is $\sqrt{97}$

$\overline{BD}$ is 5

e) P: (6.5, 2)

f) 24

(continued on T. C. 30B)

4. Plot the graphs of:

   A: (2, 0); B: (8, 0); C: (11, 4); D: (5, 4).

   Draw the figure ABCD.

   (a) What is this figure called?

   (b) Find the length of side AD.

   (c) Find the perimeter of ABCD.

   (d) Find the lengths of the diagonals (AC and BD) of ABCD.

   (e) What are the coordinates of the point of intersection of the diagonals?

   (f) What is the area of ABCD?

5. Plot the graphs of:

   A: (6, 0); B: (0, 8); C: (-6, 0).

   Draw $\triangle$ABC.

   (a) Find the perimeter of $\triangle$ABC.

   (b) Find the area of $\triangle$ABC.

   (c) Plot the graph of (0, 0). Label this point 'C'. Prove that the area of $\triangle$AOB is equal to the area of $\triangle$COB.

6. Find the length of the line segment whose end points are the graphs of:

   (a) (7, 2) and (7, 5)   (b) (5, 3) and (9, 3)

   (c) (0, 0) and (3, 4)   (d) (0, 0) and (4, 3)

   (e) (0, 0) and (-3, 4)   (f) (0, 0) and (-4, -3)

   (g) (5, 12) and (0, 0)   (h) (0, 0) and (-5, 12)

   (i) (2, 5) and (5, 9)   (j) (-3, -12) and (4, 12)

   (k) (4, 7) and (-5, 6)   (l) (8, -2) and (-6, -3)

(continued on next page)

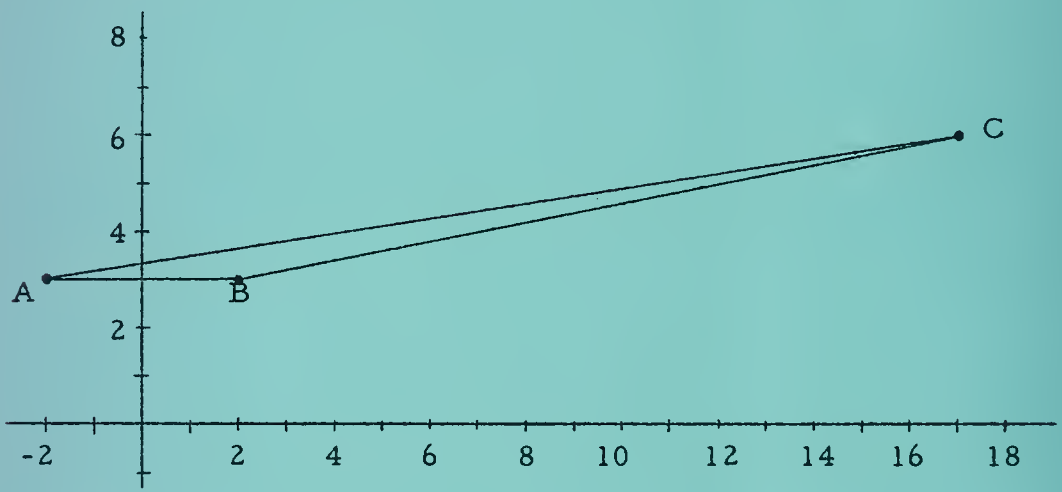
Here are supplementary exercises.

Find the area of the triangle whose vertices are

1)	A (0, 5)	B (0, 9)	C (3, 8)
2)	A (0, 5)	B (0, 9)	C (3, 17)
3)	A (0, a)	B (0, b)	C (c, d)
4)	A (a, 0)	B (b, 0)	C (c, d)
5)	A (a, 0)	B (b, e)	C (c, d)
6)	A (a, e)	B (b, e)	C (c, d)
7)	A (e, a)	B (e, b)	C (c, d)

To help students with Exercise 10, ask them to find the area of the triangle when vertices are the graphs of (-2, 3); (2, 3); and (16, 6). And (15, 6). And (14, 6). And (13, 6). And any ordered pair whose second component is 6.

10.

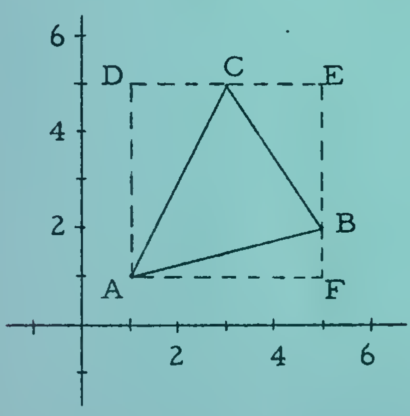


Area: 6

Perimeter: $4 + \sqrt{370} + \sqrt{234}$

Exercise 11: We hope the hint given is sufficient. Generalize this problem so that students see that you can find the area of any polygon by making appropriate arrangements of rectangles and triangles.

11.



(1) Area ΔAFB = 2

(2) Area ΔBEC = 3

(3) Area ΔCDA = 4

Sum 1, 2, 3 = 9

Area $\square$ AFED = 16

$\therefore$ Area ΔABC = 7

(continued on T. C. 31C)

Even though Exercise 7 is marked with an asterisk, we want all students to understand this formula. They should fill the blank with something like:

$$\sqrt{(|c - a|)^2 + (|d - b|)^2} \text{ ,}$$

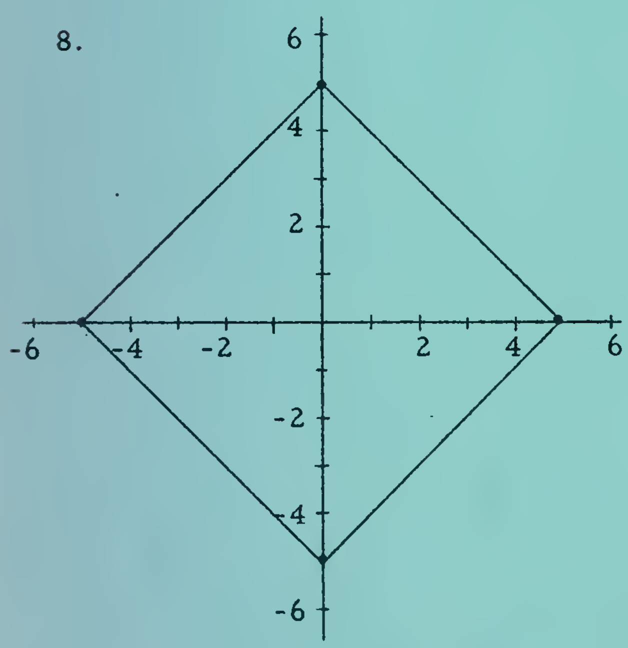
or like:

$$\sqrt{(c - a)^2 + (d - b)^2} \text{ ,}$$

and they should notice that $(a - c)^2 = (c - a)^2$ and $(b - d)^2 = (d - b)^2$, and they should be told that '$\sqrt{(c - a)^2 + (d - b)^2}$' gives the customary arrangement of letters.

* * *

8.

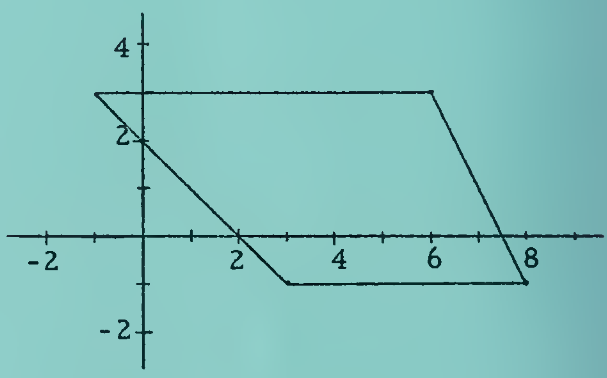


Area: 50

Perimeter: $4\sqrt{50}$

or: $20\sqrt{2}$

9.



Area: 24

Perimeter: $12 + \sqrt{32} + \sqrt{20}$

or: $12 + 4\sqrt{2} + 2\sqrt{5}$

(continued on T. C. 31B)

*7. Use your experience in finding lengths of line segments above to complete the following so that it will be a true statement.

For the graphs of every (a, b) and (c, d), the length of the line segment which connects these graphs is ______________ units.

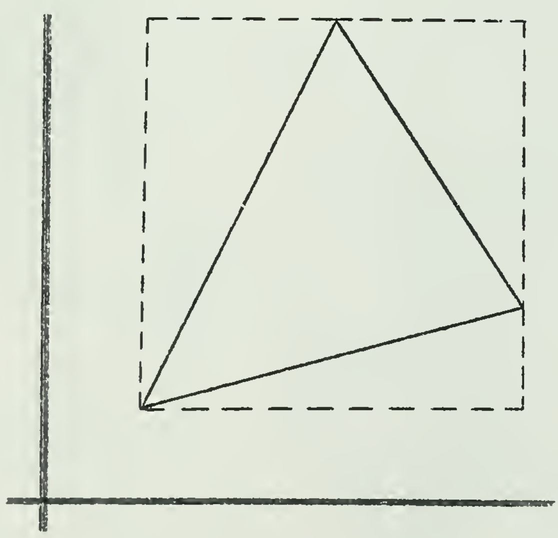
8. Find the area and the perimeter of the square whose vertices are the graphs of (-5, 0), (0, 5), (5, 0), and (0, -5).

9. Find the area and the perimeter of the polygon whose vertices are the graphs of (3, -1), (8, -1), (6, 3), and (-1, 3).

10. Find the area and the perimeter of the triangle whose vertices are the graphs of (-2, 3), (2, 3), and (17, 6).

11. Find the area of the triangle whose vertices are the graphs of (1, 1), (5, 2), and (3, 5).

<u>Hint</u>:



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4. A circular tablecloth has a diameter of 46 inches. How many yards of narrow lace will it take to go around the edge of the tablecloth?

5. How many ordered pairs of numbers are there with first components chosen from $\{3, 5, 7\}$ and with second components chosen from $\{-2, -4, -6, -8\}$? List the ordered pairs.

6. <u>A number plane game.</u>

$$(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$$

Start at (12, 18). Where are you at the end of four moves? How many moves will it take to get to (0, 0)?

Exercise 12 is strictly optional.

(b) $\frac{1}{2} \times |c - 0| \times |b - a|$

Note that the area of the triangle in (a) is not related to the replacement for 'c', and that the area of the triangle in (b) is not related to the replacement for 'd'. This is the familiar idea of the locus of points each of which can serve as the vertex of a triangle with given area and given base. [See Commentary for Exercise 10 on page 4-31.]

(c) $\frac{1}{2} \times |e - b| \times |c - a|$ (d) $\frac{1}{2} \times |d - a| \times |c - b|$

If you dare, you can ask for a formula for finding the area of a triangle none of whose sides is parallel to either axis; that is, for every a, b, c, d, e, and f, a formula for the area of the triangle whose vertices are (a, b), (c, d), and (e, f).

The formulas in Exercise 12 are important only as exercises in derivation; don't expect students to remember them.

* * *

<u>Quiz</u>.

Solve:

1. $\dfrac{n}{1\frac{2}{9}} = 3\frac{6}{13}$

2. $\dfrac{3}{a - 3} = \dfrac{5}{a + 4}$

3. Harry bought a total of 40 colored pencils at wholesale prices, and his bill was \$4.35. The pencils with red lead cost 11.25 cents each; those with blue lead cost 9.75 cents each. How many pencils with red lead did he buy?

(continued on T. C. 32B)

*12. Complete the following, making them true statements. The first statement has been completed for you.

(a) For every a, b, c, and d, the area of the triangle whose vertices are the graphs of (a, 0), (b, 0), and (c, d) is $\underline{\frac{1}{2} \times |d - 0| \times |b - a|}$ square units.

(b) For every a, b, c, and d, the area of the triangle whose vertices are the graphs of (0, a), (0, b), and (c, d) is ________________ square units.

(c) For every a, b, c, d, and e, the area of the triangle whose vertices are the graphs of (a, b), (c, b), and (d, e) is ________________ square units.

(d) For every a, b, c, d, and e, the area of the triangle whose vertices are the graphs of (a, b), (a, c), and (d, e) is ________________ square units.

C. As for plane lattices there are many sets of points on the coordinate plane. Moreover, many of these sets contain indefinitely many points so that when we indicate points in such sets by means of a diagram, we can plot only a few of the points.

Sample. Plot points in the set of all points with first coordinate equal to twice the second coordinate.

Solution. First, we find several ordered pairs in which the first number is equal to twice the second number. Examples are:

(2, 1), (3, 1.5), (-4, -2), (5, 2.5).

Then, we plot the graphs of these ordered pairs:

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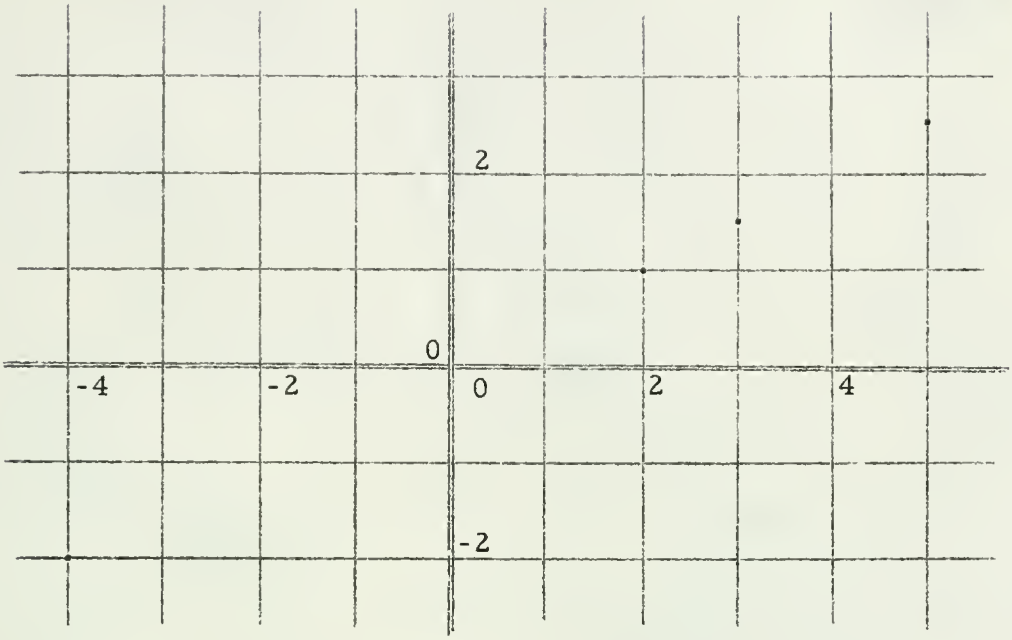


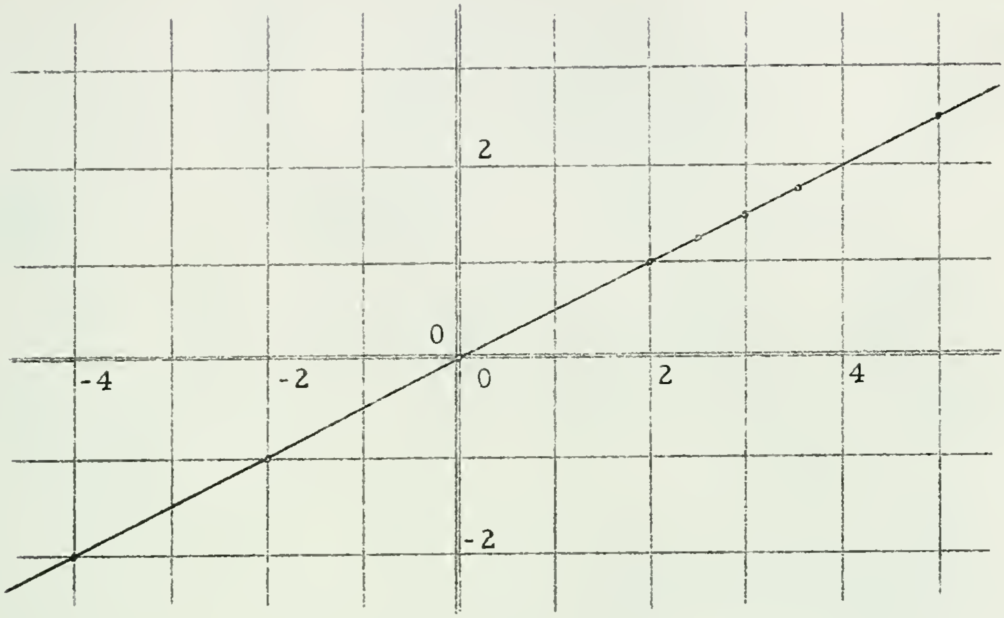
You can see that there are many more points whose coordinates are such that the first coordinate is twice the second coordinate. In fact, you can find as many of them as you please. So, you cannot plot <u>all</u> of the points in the given set. Instead, you graph enough of them so that you are sure that you know what would happen if you continued to plot more points. In this case, if you graph more ordered pairs, say, (-2, -1), (0, 0), (2.5, 1.25), and (3.6, 1.8), you will see in the following figure that all of these points appear to fall along what you think of as a straight line. You can test this idea by plotting a few more points to see if they also fall on this line. They will, and it is reasonable then for you to draw a straight line through these points and say that you have graphed the ordered pairs in the given set.

(continued on next page)

Insist that the students "check" the straight lines they draw for these exercises in the manner of parts (b) and (c) of Exercise 1. The fact that pictures of straight lines are pictures of certain kinds of sets needs to be accepted intuitively at this state. We want the two-way kind of check on intuition--points on the straight line have coordinates which fit the verbal description <u>and</u> <u>conversely.</u>

* * *

In SECOND COURSE we <u>define</u> a straight line in the number plane as the solution set of an equation in x and y of the form '$ax + by + c = 0$' where a, b, and c are real numbers and a and b are not both 0. You can tell students in FIRST COURSE that pictures of straight lines are what you get when you draw the loci of equations such as '$3x + 4y + 9 = 0$' and '$y = 7x - 2$'.



For these exercises plot enough points until you are "sure" that you know what the graph for the entire set looks like.

1. Plot points which are in the set of all points with first coordinate equal to 2 more than second coordinate.

   (a) Draw the straight line which passes through these points.

   (b) Select three ordered pairs of numbers with first number equal to 2 more than second number. Plot the graphs of these ordered pairs. Do they fall on the straight line?

   (c) Select three points on the straight line (These should be different from the points plotted at first and the points plotted in (b)). Is the first coordinate of each of these 2 more than the second coordinate?

2. Plot points which are in the set of all points with first coordinate equal to 3 more than one half of second coordinate.

3. Plot points which are in the set of all points with the sum of first and second coordinates equal to 9.

(continued on next page)

It is here that we introduce the x-axis and y-axis convention. It is essential that you impress upon the student the arbitrariness of this convention, and also its universal acceptance. Too often, we encounter students who cannot draw the locus of, say, '$a = 3b + 2$' because "it doesn't have 'x' and 'y' in it". It might be helpful to show the students this:

$$\{(y, x) : 2x + y = 7\}$$
$$\{(a, b) : 2b + a = 7\}$$
$$\{(x, y) : 2y + x = 7\}$$

and have them consider the graphs of these sets.

* * *

We are including many non-linear graphs like the one in Sample 2. You should spend a fair amount of time on Sample 2 in class, plotting many more points than we have plotted in our discussion. In addition, you should also make replacements for 'y' first and then find replacements for 'x'.

<u>Note</u>: We can abbreviate descriptions of sets of points such as those given in the preceding exercises by using pronumerals and by following a convention concerning the order of coordinates and the selection of pronumerals. We shall agree that the pronumeral 'x' is to be replaced in turn by a numeral for the first coordinate of a point and the pronumeral 'y' is to be replaced by a numeral for the second coordinate of that point. Then, for example, the phrase:

The graph of '$x = 2y$'

means the same thing as:

The set of all points with first coordinate equal to twice second coordinate.

Similarly, the phrase:

The graph of '$y = \frac{2}{3}x - 3$'

means the same thing as:

The set of all points with second coordinate equal to 3 less than two thirds of first coordinate.

Since 'x' is used for first coordinates, the first coordinate axis is often called the <u>x-axis</u> and since 'y' is used for second coordinates, the second coordinate axis is often called the <u>y-axis.</u> In diagrams of the coordinate plane, it is customary to write an 'x' next to the first coordinate axis and a 'y' next to the second coordinate axis to show that you are following the above convention.

<u>Sample 2</u>. Plot the graph of '$y \geq x + 1$'.

<u>Solution</u>. We interpret this sentence to mean that we are to plot graphs of points with second coordinate greater than or equal to 1 more than first coordinate. In other words, we are to find points whose coordinates satisfy '$y \geq x + 1$'. We do this by first replacing either 'x' or 'y' by a numeral and then by finding replacements for the other pronumeral. For example, suppose we replace 'x' by, say '1':

$$y \geq 1 + 1.$$

(continued on next page)

Now, one replacement for 'y' which satisfies:

$$y \geq 1 + 1$$

is, say, '2'. Also '3'. And '4' and '5' and '$2\frac{1}{2}$'. That is, the ordered pairs (1, 2), (1, 3), (1, 4), (1, 5) and (1, $2\frac{1}{2}$) are pairs of coordinates of points on the graph of '$y \geq x + 1$'. If we replace 'y' by a numeral for any number not less than 2, the expression '$y \geq 1 + 1$' will be satisfied. Therefore, we conclude that every ordered pair of numbers with first number 1 and second number not less than 2 will satisfy '$y \geq x + 1$'. The graphs of these ordered pairs are all the points on the same vertical grid line as the graph of (1, 2) but not below this point.

Now, replace 'x' by '$\frac{1}{2}$':

$$y \geq \frac{1}{2} + 1 .$$

Some replacements for 'y' which satisfy '$y \geq 1\frac{1}{2}$' are:

$$1\frac{1}{2}, \quad 2, \quad 3\frac{1}{2}, \quad 17, \quad 98\frac{1}{4}.$$

In fact, if 'y' is replaced by a numeral for any number which is not less than $1\frac{1}{2}$, the expression '$y \geq 1\frac{1}{2}$' will be satisfied. So, we see that every ordered pair of numbers with first number $\frac{1}{2}$ and second number not less than $1\frac{1}{2}$ will satisfy '$y \geq x + 1$'. The graphs of these ordered pairs are all the points on the same vertical grid line as the graph of ($\frac{1}{2}$, $1\frac{1}{2}$) but not below this point.

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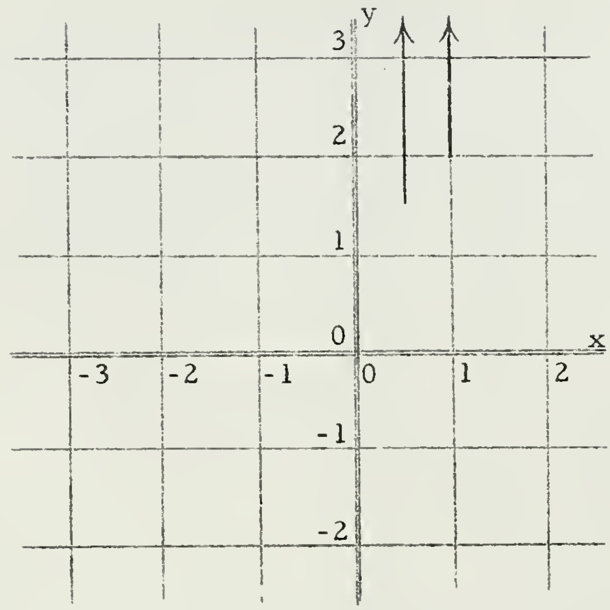

We could continue making one replacement after another for 'x' and find appropriate replacements for 'y'. When we plot the graphs of the many ordered pairs of numbers which satisfy '$y \geq x + 1$', we find these points arranged in a clear pattern on our diagram. The diagram on page 4-38 shows a region of points whose coordinates satisfy the given expression. The straight line edge of the region is included in the region because every point on the edge has coordinates which satisfy '$y \geq x + 1$'. (In fact, they satisfy '$y = x + 1$'.) The entire region is the graph of '$y \geq x + 1$'. Convince yourself that this region is the graph of '$y \geq x + 1$' by selecting points in the region (and points on the boundary, also) and points not in the region and seeing which points' coordinates satisfy '$y \geq x + 1$'.

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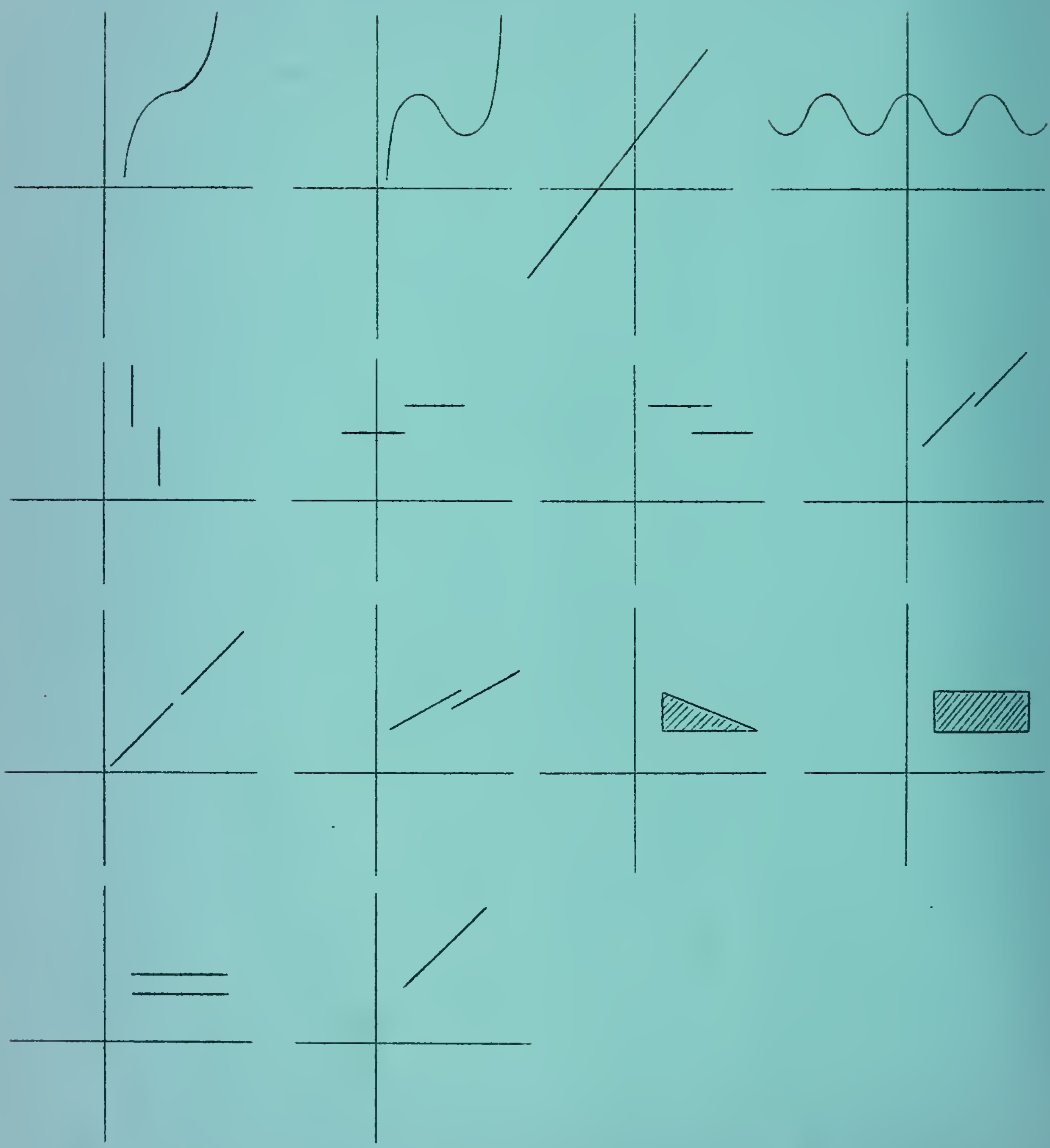


of the solution set, th[illegible] ordered p[illegible] th[illegible] first component and a [illegible] second component could [illegible] only be an element [illegible]

[illegible] which could b[illegible] satisfied [illegible] have the same first component. [illegible]

[illegible] satisfied by ordered pairs [illegible] the same second component [illegible] $y = 2$ and $y = 0x + 1$

[illegible] should b[illegible] consider [illegible] three types of equations and their loci:

a) those satisfied by ordered p[illegible] of which ha[illegible] the same first component;

b) those satisfied by ordered p[illegible] which have the same second component;

c) those satisfied by ordered pa[illegible] of which have the same first component or the same second component.

Ask the students to identify the type illustrated by these [illegible]

of the solution set, then an ordered pair with the same first component and a different second component could not possibly be an element of the solution set for <u>this</u> equation!

Then ask the students to give some equations which could be satisfied by ordered pairs of numbers all of which have the same first component. Examples are: $x = 0y$ and: $x = 3$.

Next consider equations which are satisfied by ordered pairs having the same second component. Examples: $y = 2$ and: $y = 0x + 117$.

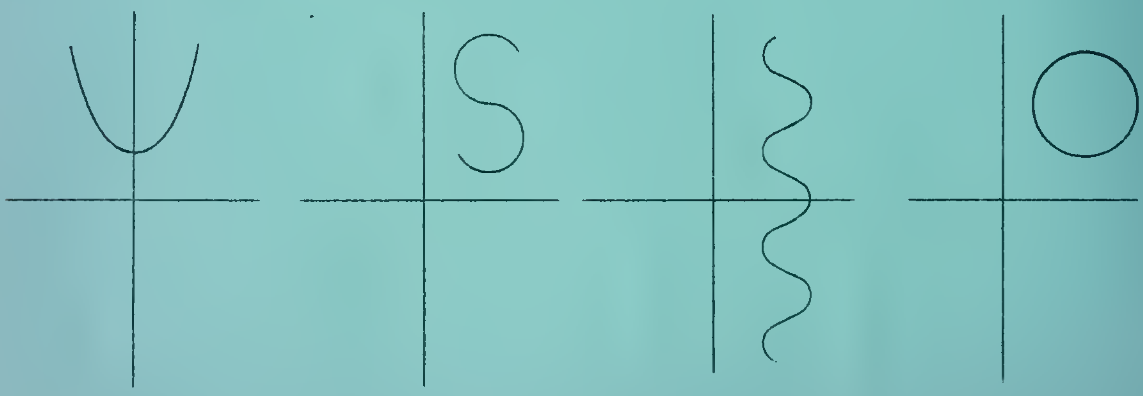
Now students should be ready to consider these three types of equations, and their loci:

a) those satisfied by ordered pairs some of which have the same first component;

b) those satisfied by ordered pairs some of which have the same second component;

c) those satisfied by ordered pairs none of which have the same first component or the same second component.

Ask the students to identify the type illustrated by these graphs:



(continued on T. C. 38C)

We have not shown the so-called "table of values" device. We think this device is related more to the "work habits" of the student than to the mathematics of the problem. If you do want your students to list ordered pairs in a table, we recommend a vertical arrangement with first coordinate given in the left-hand column. A "table of values" loses much of whatever effectiveness it may have in the case of non-linear graphs such as those we are working with. [Incidentally, you should encounter some quizzical looks if you use the word 'values'. We introduce this word on page 4-44.]

* * *

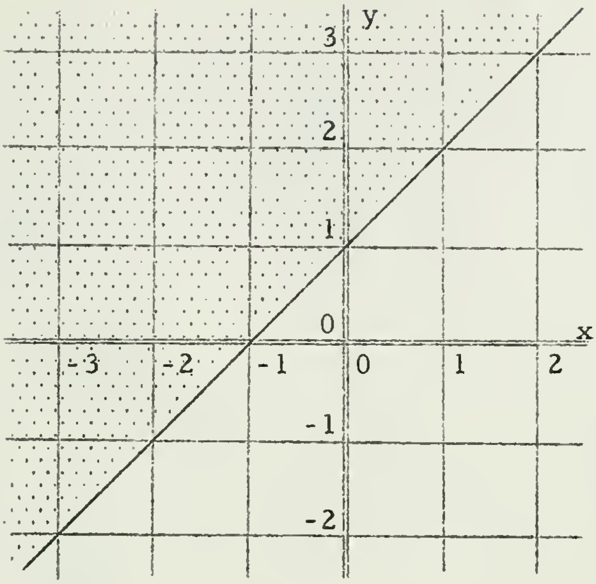
As you discuss these exercises, use questions to help the students understand that the locus of an equation [or inequation] contains those points, and only those points, whose coordinates satisfy the given equation [or inequation]. That is:

(i) all ordered pairs which are elements of the solution set of the given sentence are points which are in the locus, and

(ii) all ordered pairs which are <u>not</u> elements of the solution set of the given sentence are points which are <u>not</u> in the locus.

Consider the equation '$y = \frac{1}{3}x + 12$'. Ask students whether (300, 112) is an element of the solution set. If the reply is 'yes', then the point (300, 112) is in the locus of the given equation. Next consider (270, 102); (1530, 522); (612, 214). When someone explains that (612, 214) is not an element of the solution set, ask where the point (612, 214) would be in relation to the locus of the given equation. Then ask about (300, 9720). An orchid to the student who explains that since they have already agreed that (300, 112) <u>is</u> an element

(continued on T. C. 38B)



4. Plot the graph of '$y \leq 3x$'.

5. Plot the graph of '$y \geq x - 2$'.

6. Plot the graph of '$y = \frac{1}{2}x + 4$'.

7. Plot the graph of '$y = 3$'.

8. Plot the graph of '$x \geq -3$'.

9. Plot the graph of '$x + y \leq 1$'.

<u>Note</u>: We can abbreviate:

Plot the graph of

as:

Graph.

The word 'graph' is then used as a verb.

<u>Graph each of the following</u>:

10. $3x = y$

11. $y = x + 1$

12. $x + y = 0$

13. $2x + 3 = y$

14. $y = x - 3$

15. $x = 2$

16. $y = -5$

17. $10x + 4y = 0$

(continued on next page)

7. Below are listed pairs of numbers. Insert '>' or '<' between the two numerals in a pair so that the resulting statement is true.

a) $-\frac{5}{8}$ $\quad -\frac{5}{9}$ b) $\frac{5}{6}$ $\quad \frac{7}{8}$ c) $\frac{4}{5}$ $\quad -\frac{14}{15}$

d) $\frac{8}{9}$ $\quad \frac{10}{11}$ e) $-\frac{187}{452}$ $\quad -\frac{186}{453}$

* * *

Extra credit problem.

Rule for a complete number plane game:

$$(x, y) \rightarrow (\frac{x}{2}, \frac{y}{2})$$

1. Start at (24, 24). Where are you at the end of 1 move? 2 moves? 3 moves? 4 moves?

2. How many moves will take you from (24, 24) to (0, 0)?

3. What is the smallest number of moves required to take you from (24, 24) to a point inside of a circle with center at (0, 0) and radius 1? Radius 0.01? Radius 0.00001?

How many points are there in each of these sets?

g) $\{(x, y): x = y + 2\}$

h) $\{(x, y): x = 2y\}$

i) $\{(x, y): y = 3x - 2\}$

j) $\{(x, y): x = 2y - 1\}$

2. When you are throwing a green die and a white die, and agree that the green die gives the first number of an ordered pair while the white die gives the second number of an ordered pair, what are the chances that when you make a throw you will get:

a) the pair (3, 2)?

b) the pair (5, 6)?

c) the pair (2, 5) or the pair (1, 4)?

3. The expression '$(a - 7)(a + 9)$' is equivalent to which of the following?

a) $aa - 63$

b) $aa + 2$

c) $aa + 2a - 63$

d) $aa - 2a - 63$

e) $aa - 16a - 63$

4. In a certain plane lattice game, moves are made according to the following rule:

$$(p, q) \rightarrow (-p, -3q) .$$

Start at the graph of $(-3, 5)$. At the end of <u>three</u> moves you should be at the graph of which of the following ordered pairs?

a) $(-3, 135)$

b) $(3, 135)$

c) $(-3, +15)$

d) $(+3, -45)$

e) $(+3, -135)$

5. Which of the following equations has roots +1 and -1?

a) $7.5bb + 7.5 = 0$

b) $3bb - 3 = 0$

c) $\frac{1}{5}aa = \frac{1}{25}$

d) $.5xx = 0$

e) $\frac{1}{4}nn = 4$

6. The bases of a trapezoid are 8 inches and 9 inches, respectively. The height of the trapezoid is 3t inches. What is the area of the trapezoid?

(continued on T. C. 39C)

In Exercise 24, students can indicate that the two parallel boundaries are not in the graph by drawing them as dotted lines.

* * *

In Exercises 28 through 33 note again the importance in distinguishing between the use of 'and' and the use of 'or'. For example, the graph in Exercise 32 consists of 3 parallel horizontal lines, and the graph in Exercise 33 is the empty set.

* * *

Beware of division by 0 in Exercises 37 and 38.

* * *

For more exercises like those in Part C and for a discussion of some of the pedagogical problems involved in graphing equations and inequations, see "Graphing in Elementary Algebra" by Beberman and Meserve in the April, 1956 issue of <u>The Mathematics Teacher</u>. You will notice that this article is directed to teachers of conventional courses, and, therefore, uses a somewhat different terminology from ours.

* * *

Here are supplementary exercises for Part C.

(1) $x = 3$ and $|y| \geq 3$ (2) $x = y$ and $x \geq 4$

(3) $|x| \leq 3$ and $|y| \geq 2$ (4) $|x| \geq 10$ and $|y| \geq 10$

* * *

<u>Quiz</u>.

1. On a lattice plane, with axes labelled so that for every ordered pair of whole numbers (x, y), $-4 < x < 4$ and $-3 \leq y \leq 3$ if and only if there is a point which is the graph of (x, y). On such a chart, how many points are there with:

   a) first component -2? b) second component $+3$?

   c) first component ≥ 3? d) second component ≤ -3?

   e) first component > 2 <u>and</u> second component < 3?

   f) first component = second component?

(continued on T. C. 39B)

18. $y = 5x$

19. $3y = 4x + 1$

20. $3y + 2x = 6$

21. $3y - 2x = 6$

22. $xx = 9$

23. $xx \leq 9$

24. $|y| < 3$

25. $|xy| > 0$

26. $x + y = 7 + x$

27. $xy = 0$

28. $x = 3$ and $y = 2$

29. $x = 3$ or $y = 2$

30. $x = 3$ and $y > 2$

31. $x = 3$ or $y > 2$

32. $y = 7$ or $y = 5$ or $y = 3$

33. $y = 7$ and $y = 5$ and $y = 3$

34. $x + y = y + x$

35. $x - y = y - x$

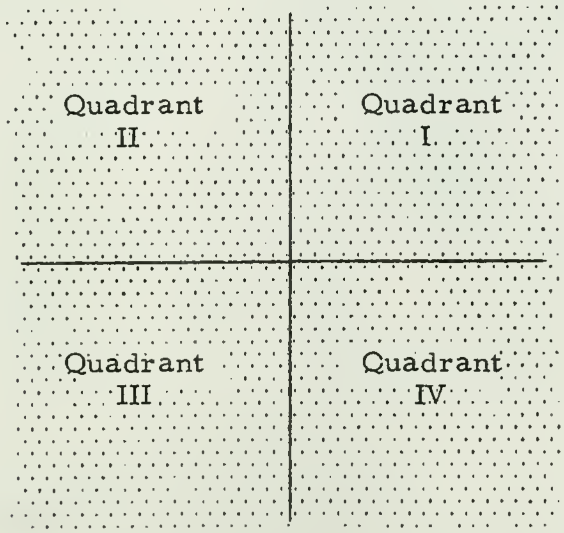
36. $xy = yx$

37. $x \div y = y \div x$

38. $\frac{x}{x} + \frac{y}{y} = 2$

D. The two coordinate axes separate the points in the coordinate plane into four regions called <u>quadrants</u>. Each quadrant is a set of points.

Quadrant I is the graph of '$x > 0$ and $y > 0$'.
Quadrant II is the graph of '$x < 0$ and $y > 0$'.
Quadrant III is the graph of '$x < 0$ and $y < 0$'.
Quadrant IV is the graph of '$x > 0$ and $y < 0$'.



[illegible] direct [illegible] [illegible] the student will make [illegible] [illegible] of results in the earlier equations [illegible] should [illegible] direction of [illegible] each straight [illegible] intersection with [illegible]

[illegible] After the students have [illegible] questions to determine [illegible] "slope" and "y intercept" [illegible] determined [illegible] formation of the [illegible]

[illegible] locus of each equation in these [illegible] what do you notice [illegible]

[illegible] $y = x + 6$ [illegible]

The quadrant exercises are designed to give a student a feeling for the relation between direction of slant of a straight line and the quadrants it intersects. We suspect the student will make a series of rapid replacements for 'x' in, say, Exercise 1(b), to determine the existence of points in the various quadrants. He should be able to predict the direction of slant of each straight line by noting its intersection with each of the various quadrants.

* * *

Here are supplementary exercises to use. After the students have drawn the loci of these equations, use questions to determine whether they have discovered the idea of "slope" and "y-intercept" which can be determined by examination of the equation.

Draw the locus of each equation in these pairs. What do you notice about the pair of loci?

a) $y = -3x + 2$
   $y = 3x + 2$

b) $y = \frac{1}{2}x - 5$
   $y = -\frac{1}{2}x - 5$

c) $y = x + 6$
   $y = -x + 6$

d) $y = 2x - 4$
   $y = 2x + 4$

e) $y = -\frac{3}{2}x + 3$
   $y = -\frac{3}{2}x - 3$

f) $y = x - 10$
   $y = -x + 10$

g) $x = y + 7$
   $x = -y + 7$

h) $x = -2y + 5$
   $x = 2y + 5$

i) $x = \frac{1}{3}x - 6$
   $x = -\frac{1}{3}x - 6$

j) $x = 3y + 4$
   $x = 3y - 4$

k) $x = y + 10$
   $x = y - 10$

l) $x = \frac{5}{4}y - 8$
   $x = \frac{5}{4}y + 8$

What is the union of the four quadrants and the coordinate axes? What is the intersection of any two quadrants? What is the inter-section of any quadrant and an axis?

Look at the following table. For each of the expressions in the left-hand column, tell which quadrants its graph intersects. You may not have to draw a graph of an expression to answer this question. [The first exercise has been completed for you. Check it.]

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
1. (a) $y = -2x + 3$	Yes	Yes	No	Yes
(b) $y = -x + 3$				
(c) $y = 3$				
(d) $y = x + 3$				
(e) $y = 2x + 3$				
2. (a) $y = -2x - 3$				
(b) $y = -x - 3$				
(c) $y = -3$				
(d) $y = x - 3$				
(e) $y = 2x - 3$				
3. (a) $y = 0$				
(b) $x = 0$				
4. (a) $x = 4$				
(b) $x = y + 4$				
(c) $x = -y + 4$				

(continued on next page)

Draw the loci of the following equations :

(1) $y > x$

(2) $x + y = y + x$

(3) $\frac{x}{y} = 1$

(4) $x > 2$ and $y > 3$

(5) $|x + y| < 0$

(6) $|x + y| > 0$

(7) $x < -4$ or $y > +15$

(8) $x = y$

(9) $x = -5$ and $y < 0$

(10) $x > 0$ and $y < 10$

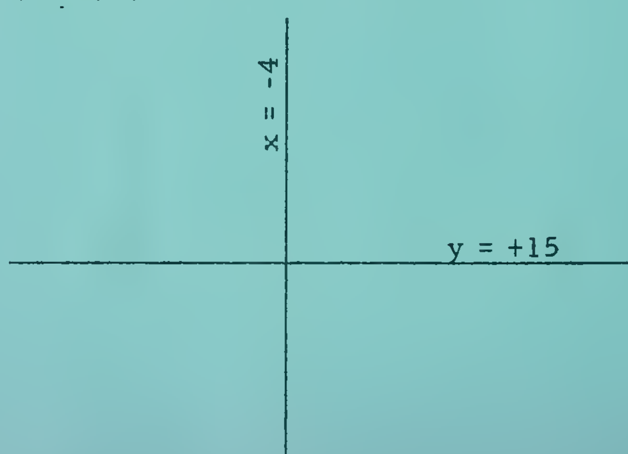
* * *

Many of your students will have already developed short-cuts and rapid methods for graphing linear equations but here is a place to compare notes and to be sure that all of them are aware of some of the "slicker" methods. They should realize that if they replace 'x' by '0' and then solve the equation for 'y', they have a rapid method for finding the coordinates of that point on the y-axis which is also a point on the locus in question. Similarly, if they replace 'y' by '0', and solve the resulting equation for 'x', they have a rapid method for finding the coordinates of the point where the line crosses the x-axis. Finding these two points is enough to allow them to sketch the line unless the points are very close to the origin. Theoretically, of course, as long as the points are different, there is one and only one line passing through both of them, but in practice, when the paper-distance between the two points is small, it is difficult for the student to draw accurately a line through the points. In the case where the line goes through the origin, then one other point <u>must</u> be found. Students should be encouraged to draw their line after having found two points on it and then to check the possibility of an error in the work by plotting a third point. If this third point falls on the line (or reasonably close to it), then they can be confident that they have the correct graph. You may also want to establish the "ground rule" of writing the equation next to the picture of its graph.

* * *

Here are supplementary exercises.

Suppose you had grid paper labelled like this:



(continued on T. C. 41C)

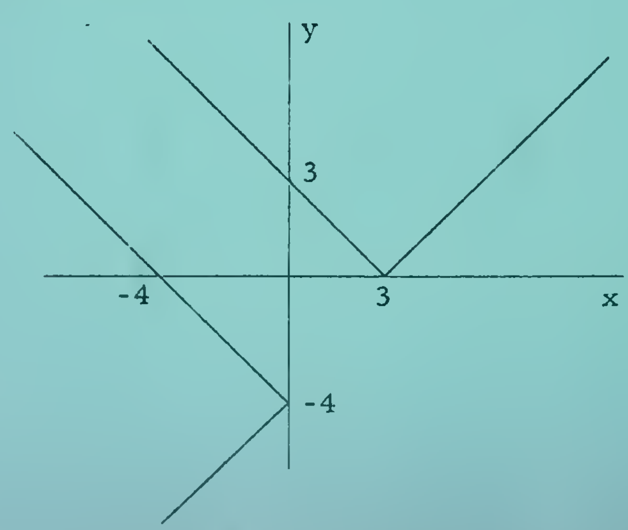
Some of your students will recognize by now that the locus of the equation in Exercise 6 is a circle with radius 5 and center at the origin. Of course, the students need only recognize that the graph has a point in each quadrant.

* * *

Students should actually make drawings for a considerable number of these exercises. However, some students may discover algebraic or arithmetic techniques for finding the coordinates of the points of intersection without graphing. Such personal exploration should not be discouraged but do not require the entire class to consider algebraic solutions of these systems of equations. Let those students who are interested hunt on their own for algebraic methods. In SECOND COURSE, there is a complete treatment on solution of systems of linear equations in two pronumerals.

Each equation in these exercises except Exercises 7, 8, 11, and 12 has a graph which is a straight line. Students will probably come to the conclusion that <u>two</u> straight lines either intersect in the empty set, or intersect in the set consisting of a single point.

The equations in Exercise 12 may give students some difficulty. Rather than looking for formal methods, students should be encouraged to hunt around until they find enough points to see what the loci of these equations look like. The loci do not intersect. Here is a sketch of the two loci:



(continued on T. C. 41B)

	Quadrant			
	I	II	III	IV
5. (a) $x = -4$				
(b) $x = y - 4$				
(c) $x = -y - 4$				
6. $xx + yy = 25$				
7. $\|x\| = 1$ and $\|y\| = 1$				

E. Each of the following exercises gives a pair of equations. Using the convention from Part C about the x-axis and the y-axis, graph each of the two equations and tell the coordinates of the points in the intersection of the graphs.

1. (a) $x + y = 6$
   (b) $3x + y = 2$

2. (a) $2x + 5y = -5$
   (b) $y + 3x = 12$

3. (a) $5x = 3 - 2y$
   (b) $y = 5 + x$

4. (a) $x = y$
   (b) $2x = 3 + 3y$

5. (a) $x = 7$
   (b) $y + 8 = 2x$

6. (a) $y = 3$
   (b) $2x = 7 + x$

7. (a) $|x| = 3$
   (b) $y - 2x = 5$

8. (a) $yy = 25$
   (b) $xx = 36$

9. (a) $2x = 3y$
   (b) $2x = 3y + 5$

10. (a) $5x - 2y = 8$
    (b) $6(y + 4) = 15x$

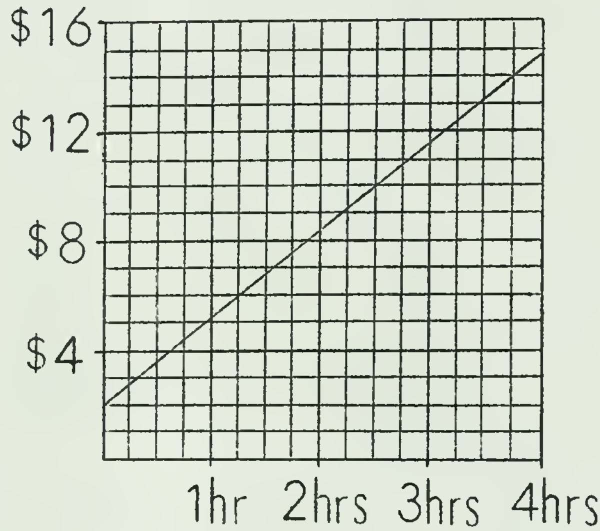
11. (a) $|x| = 5$
    (b) $|y| = 4$

12. (a) $|x - 3| = y$
    (b) $|y + 4| = -x$

Some students may object that the repairman would not keep track of his time to the nearest 15 minutes. This may be the case, but it should not concern us here. The idea is to learn to use a straight line graph in this type of problem. There may be some disagreement in (5) over how long the repairman worked if his charge was $2.00. The idea we had in mind was that he was able to make the repair almost immediately, and that he did not charge for any working time but merely for making the call. An impossible problem is given in (6). We give more attention to the question of "permissible" problems later in this section.

GRAPHS OF FORMULAS

A repairman uses the following procedure in charging for service calls. He charges $2.00 for going to a home and he charges $3.20 more for each hour that he works. He could use the following chart to determine the amount to charge.



Use the chart to answer these questions:

(1) What is the charge if the repairman works 2 hours?

(2) What is the charge if the repairman works $3\frac{1}{2}$ hours?

(3) What is the charge if the repairman works 45 minutes?

(4) How long did the repairman work if the charge was $6.25?

(5) How long did the repairman work if the charge was $2.00?

(6) How long did the repairman work if the charge was $1.00?

The repairman could also use a formula instead of a chart to compute his charge. If we use 'c' to hold a place for a numeral which tells his charge and 't' to hold a place for a numeral for the number of hours he works, then the equation:

$$c = 2.00 + 3.20t$$

gives a true statement which tells his charge for each appropriate

There is an important idea here which should be stressed. In the chart on the previous page the restrictions imposed by the practical aspects of the problem were apparent from looking at the chart. So, the student could tell immediately that there could be no charge less than $2.00, and that the only kinds of working time contemplated were positive numbers of hours. However, if he uses the corresponding equation on this page, and if he ignores the practical nature of the problem, he can make replacements in the equation using any kind of numbers at all, positive, negative, or zero. Thus, in answering the question in (11), he could replace 'c' by '.40', and solve the resulting equation for 't'. He would obtain a negative number. His arithmetic and algebra would have been formally correct, but he has not considered the original restrictions imposed by the problem.

replacement of 'c' and 't'. For example, if he worked 2 hours, we replace 't' by '2' and obtain:

$$\begin{aligned} c &= 2.00 + 3.20(2) \\ &= 2.00 + 6.40 \\ &= 8.40 \end{aligned}$$

which tells us that 'c' should be replaced by '8.40'. Therefore, his charge for working 2 hours is \$8.40.

Use the equation:

$$c = 2.00 + 3.20t$$

to answer these questions:

(7) What is the charge if the repairman works 4 hours?

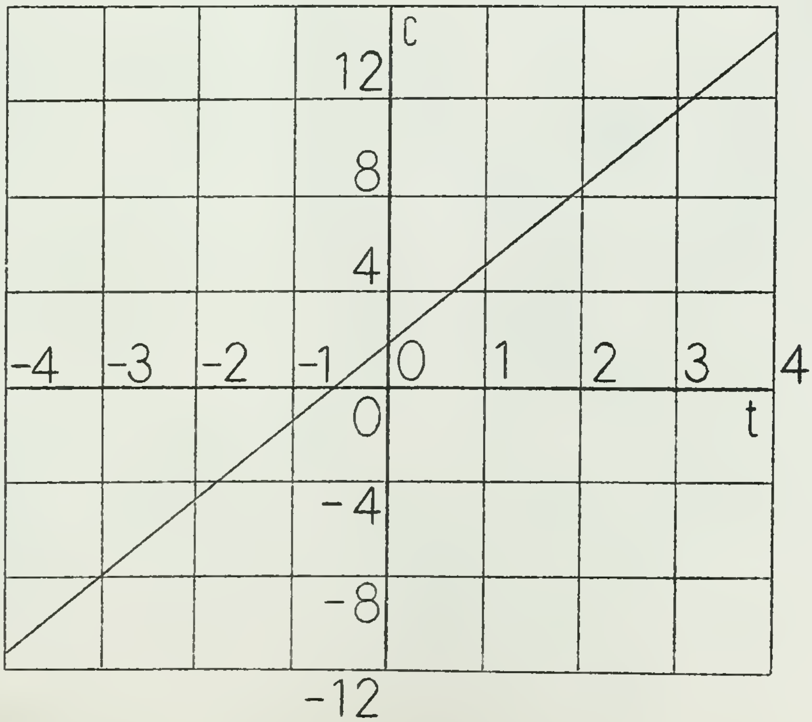
(8) What is the charge if the repairman works $2\frac{1}{2}$ hours?

(9) What is the charge if the repairman works 75 minutes?

(10) How long did the repairman work if the charge was \$2.80?

(11) How long did the repairman work if the charge was \$.40?

When you used the chart before to find charges, you were using the graph of the equation 'c = 2.00 + 3.20t'. If you were told to graph 'c = 2.00 + 3.20t' (using a c-axis for the vertical axis and a t-axis for the horizontal axis), you would get something like this:



Here for the first time we introduce the technical use of the word 'value' into the course. This is a much-abused word. We are using it in a rather special way. Whenever you use a pronumeral, there must always be in the background the set of numbers that can be used in making replacements for that pronumeral. When we make a replacement for a pronumeral, we write the name of a number in place of the pronumeral. Thus, in discussing the numbers which can be used in making replacements, we frequently find ourselves saying "a number whose name may be put in place of the pronumeral". In order to shorten our language, we call such a number a <u>value of the pronumeral</u>. Thus, for the pronumeral 'c' on page 4-43, 3 is a value of 'c' but 1 cannot be a value of 'c'. Similarly, 5 is a value of 't' but -1 is not a value of 't'. Students should realize that they cannot tell which numbers are values of a pronumeral by looking at the pronumeral; rather they must look at the problem or context in which the pronumeral is used. In many of the problems in this unit such as, say, the problem of graphing the equation 'x + 2y = 10', each directed number is a value of either of the two pronumerals used.

* * *

Often, the word 'range' is used instead of the word 'domain'. We have chosen the term 'domain' here because it is a better kind of groundwork for the use of 'domain' and 'range' as technical terms when working with functions. Be sure that students understand that the <u>domain of a pronumeral</u> is a set of numbers. Of course, if you followed our suggestion about introducing 'domain' in Unit 2, the discussion on page 4-44 will serve as review.

This graph contains points corresponding to some ordered pairs which satisfy the equation but which do not make sense in the problem to which the equation applies. For example, points in the second quadrant would correspond to negative numbers of hours worked. Points in the third quadrant would correspond to negative numbers of dollars charged and negative numbers of hours worked. Clearly, Quadrants II, III, and IV do not apply in this problem. Therefore, we should show only the points in Quadrant I when making a graph of the formula '$c = 2.00 + 3.20t$' for finding repair charges.

Whenever you make a graph for a formula, you should keep in mind the sets of numbers whose numerals are allowed to be substituted for the pronumerals in the formula. The ordered pairs used in making the graphs should contain numbers from these sets only. Of course, if you do not know that an equation is to be used as a formula, you graph it as you would any other equation without restrictions on the numbers which can be used.

When using pronumerals in formulas, it becomes tiresome to repeat frequently "A number whose numeral may be substituted for the pronumeral". We shall abbreviate this expression by calling such a number a value of the pronumeral. Thus, if the numeral '3' may be substituted for 't' in a formula, the number 3 is a value of the pronumeral 't'. The set of all values of a pronumeral in a particular formula is called the domain of that pronumeral for the problem. In the formula '$c = 2.00 \div 3.20t$' above, the domain of 't' is the set of all positive numbers and the domain of 'c' is the set of all positive numbers greater than 2. Explain.

EXERCISES

1. A formula for finding an approximate number of inches in the circumference of a circle is:

$$C \overset{a}{=} 6.28R, \quad R \geq 0.$$

The restriction '$R > 0$' means that the formula should be used only when 'R' is replaced by numerals for numbers of arithmetic

(continued on next page)

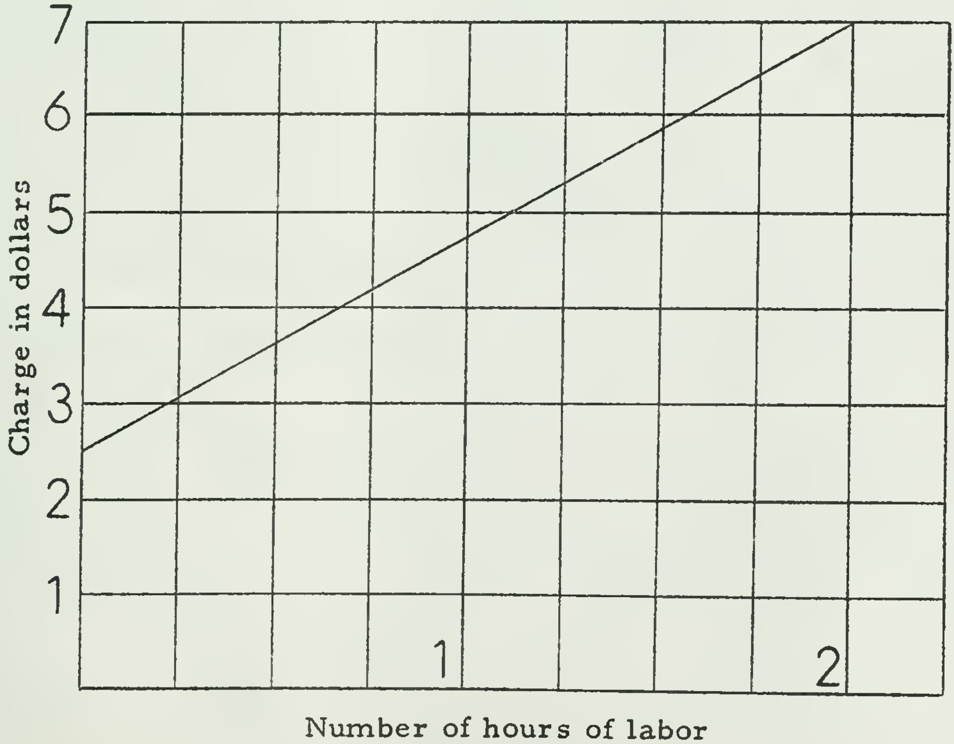
In Exercise 3 students should make a graph which is a set of discrete points; they should indicate the domains of their pronumerals accordingly. Thus, the domain of the pronumeral for the number of towels is the set of whole numbers greater than 0; that is, $\{n: \text{ n is a whole number and } n > 0\}$. The domain of the pronumeral for the charges is the set of numbers 1, 1.02, 1.04, etc; that is, $\{c: \text{ there is a whole number } n \geq 0,\ c = 1 + .02n\}$. [Although people in the towel service business are much concerned with the upper bounds of these domains (as well as with the lower bounds), we are not concerned with the upper bounds, and assume here that there is no upper bound.]

(or positive numbers and 0). That is, the domain of 'R' is the set of all numbers greater than or equal to zero. What is the domain of 'C'? Make a graph for this formula. Be sure that you label each axis.

2. Each trapezoid in a set of trapezoids has a 10-inch base and an 18-inch base. Give a formula for finding the area of each trapezoid in the set when you know its altitude. Make a graph for this formula. Label the axes. Tell the domains of your pronumerals.

3. A "Towel Service" charges \$1.00 minimum per week for making two calls and for the use of a container. In addition to the minimum, it charges \$.02 for each clean towel supplied. Give a formula for finding the weekly charge in terms of the number of towels used. Tell the domain of each of your pronumerals. Make a graph for this formula. (Hint: Do not make a graph which tells the charge for, say, $5\frac{1}{2}$ towels.)

4. A repairman uses the following chart to determine charges for his services:



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Notice that this sequence of questions for Exercise 4 leads to finding the equation which has a given locus. Students should not have much difficulty in this exercise because of the sequential nature of the questions, and because of their earlier experience with the repairman-formula. However, if your class has time for some side excursions, it might be instructive to draw a variety of graphs on the board and ask them to find the corresponding equations which have these graphs. Some of your students will begin to find systematic methods for finding these equations.

* * *

In Exercise 5 students are given the domain of 'C'. After they have answered the questions and become familiar with the formula, you might ask them for the domain of 'F' which corresponds to the given domain of 'C'. The correct answer is the set of numbers between 212 and -148; that is, $\{F: -148 \leq F \leq 212\}$.

If your class is interested and has the time, you might bring a Centigrade and a Fahrenheit thermometer to class or refer students to encyclopedia discussions of these scales and other temperature scales such as the Kelvin scale and the Réaumur scale.

Also, you may want the class to consider whether there exists a temperature for which the readings are the same on both Centigrade scale and Fahrenheit scale. Ask how they could find the number which represents such a temperature reading if such exists. [Replace 'C' by an 'F' in the equation '$F = \frac{9}{5}C + 32$', or find the point in the intersection of the locus of '$F = \frac{9}{5}C + 32$' and the locus of '$F = C$'.]

(a) How much does he charge if he just makes the call and doesn't count any time at all?

(b) How much does he charge if he works one hour?

(c) How much does he charge if he works two hours?

(d) Not counting the charge for making the call, what is his charge per hour for labor?

(e) Give a formula which corresponds to the graph. Tell the domains of the pronumerals in the formula.

5. Temperature may be measured on a Fahrenheit scale or on a centigrade scale. Thus, the temperature of boiling water may be given as 212° F. or 100° C.. A formula for finding the Fahrenheit reading when you know the centigrade reading is:

$$F = \frac{9}{5}C + 32.$$

Make a graph for this formula using the restriction '$-100 \le C \le 100$', and answer the following questions:

(a) What Fahrenheit reading corresponds to a centigrade reading of 0°?

(b) If the Fahrenheit reading is 80°, what is the centigrade reading?

(c) If the centigrade reading is -15°, what is the Fahrenheit reading?

(d) If there is a 10 degree increase in the temperature measured on the centigrade scale, what is the corresponding increase measured on the Fahrenheit scale?

(e) If there is a 20 degree decrease on the Fahrenheit scale, what is the corresponding decrease on the centigrade scale?

(f) Suppose the out-of-doors temperature rises during the period 12:00 noon to 1:00 p.m. on a certain day. Which of the two scales will show a greater numerical difference?

(g) Show how you could obtain the above formula by using the following facts: (1) the temperature of boiling water is 100° C. or 212° F., and (2) the temperature at which water freezes is 0° C. or 32° F.

(continued on next page)

We have been considering the introduction of this symbol, '$[\![- - -]\!]$', in the text and giving some exercises which make use of it. The use of this symbol opens up a whole new field of interesting and highly challenging exercises. Introduce this symbol to your students by giving them the postage formula above and then give them practice in using the symbol. Try exercises like these:

A. Simplify.

1. $[\![3.6]\!]$ 2. $[\![7.6]\!] + [\![1.3]\!]$ 3. $[\![9.2 + 3.9]\!]$

B. True or false?

1. For every x, $[\![x + 1]\!] = [\![x]\!] + 1$.
2. For every x, $[\![x + 2]\!] \geq [\![x]\!] + 2$.
3. For every x and y, $[\![x + y]\!] \leq [\![x]\!] + [\![y]\!]$.
4. For every x and y, $[\![xy]\!] \geq [\![x]\!] \times [\![y]\!]$.
5. For every x, $|[\![x]\!]| = [\![|x|]\!]$.
6. For every x, $[\![x]\!] - [\![x - 0.5]\!] = 0$ or 1.

C. Solve these equations.

1. $[\![x]\!] = 2$ 2. $[\![x]\!] = 1.7$

3. $[\![x - 1]\!] = [\![x]\!] - 1$ 4. $2 - [\![x]\!] = 4 \times [\![x]\!]$

D. Graph.

1. $y = [\![x]\!]$ 2. $[\![y]\!] = [\![x]\!]$

The symbol '$[\![x]\!]$' is correctly pronounced as 'the greatest whole number not greater than x'. Your students may quickly tire of this lengthy expression and invent a shorter one such as 'brack x'. Encourage them to do so.

* * *

If one were to look at the chart on page 4-47 without any knowledge of the postal system, one would wonder what the charge was for exactly 1 ounce [or exactly 2 ounces, etc.], since it appears that both 3 and 6 correspond to 1. The small dots [they should be larger or darker for better visibility] on the graph are intended to settle this question. Thus, since the small dot above the '1' is at '3', the student should realize that the chart tells him that the charge for 1 ounce is 3 cents.

Some of your students who look for linguistic niceties may find fault with the wording, '3 cents for each ounce or part of an ounce' [which is adapted from the post office's actual wording, '3 cents an ounce or fraction thereof']. One might try to make a strict application of this rule as follows: A letter weighs 1 ounce. An ounce is $\frac{3}{3}$ of an ounce. Thus, there are three parts of an ounce in an ounce. But, since there is a charge of 3 cents for each part of an ounce, the postage charge is 9 cents. By using this strict interpretation, the charge can be computed to be as large as you please. Of course, this is not what the post office means. The mathematician's method to avoid this ambiguity would be to proceed as follows: Weigh the letter and find the number of ounces. Now, find the smallest whole number which is greater than or equal to that number of ounces. Multiply this whole number by 3 to obtain the number of cents in the postage charge.

* * *

Mathematicians have a symbol which can be used in writing formulas for problems such as this one. They write heavy or boldface brackets around a numeral or pronumeral and define the new symbol as follows:

For every x, $[\![x]\!]$ is the greatest whole number not greater than x.

Thus,

$$[\![2\tfrac{1}{3}]\!] = 2 \qquad [\![-6.7]\!] = -7 \qquad [\![\tfrac{1}{2}]\!] = 0$$

$$[\![4]\!] = 4 \qquad [\![5.999]\!] = 5 \qquad [\![105\%]\!] = 1$$

$$[\![4.7]\!] + [\![2.4]\!] = 4 + 2 \neq [\![4.7 + 2.4]\!] = 7$$

$$[\![3.2]\!] + [\![4.1]\!] = 3 + 4 = [\![3.2 + 4.1]\!]$$

One possible first-class-mail formula could be written as follows:

$$P = 3 \times |[\![-W]\!]|.$$

The domain of 'P' is the set of all positive multiples of 3, and the domain of 'W' is the set of all positive numbers.

* * *

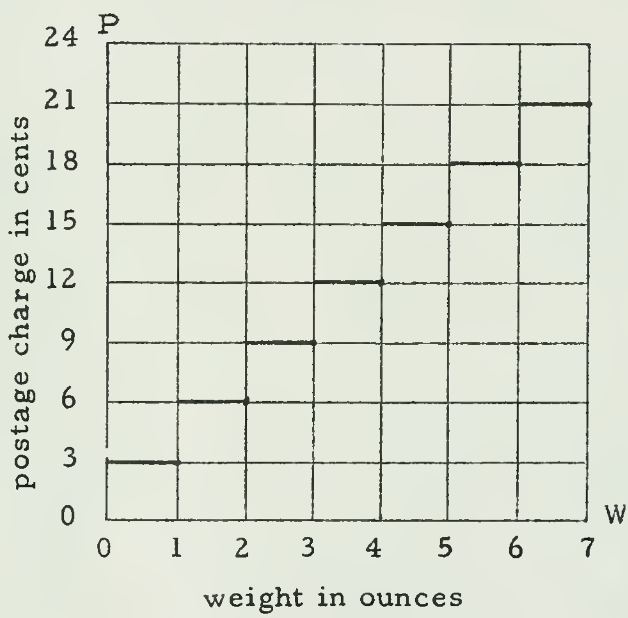
(continued on T. C. 47B)

6. The postage charge on first class mail is "3 cents for each ounce or part of an ounce." Use the following chart to find the postage charge for

(a) $2\frac{1}{2}$ ounces (b) $\frac{1}{8}$ ounce

(c) $5\frac{1}{4}$ ounces (d) $\frac{1}{10}$ ounce

[Note: The graph shown in this chart is often called a step-graph.]



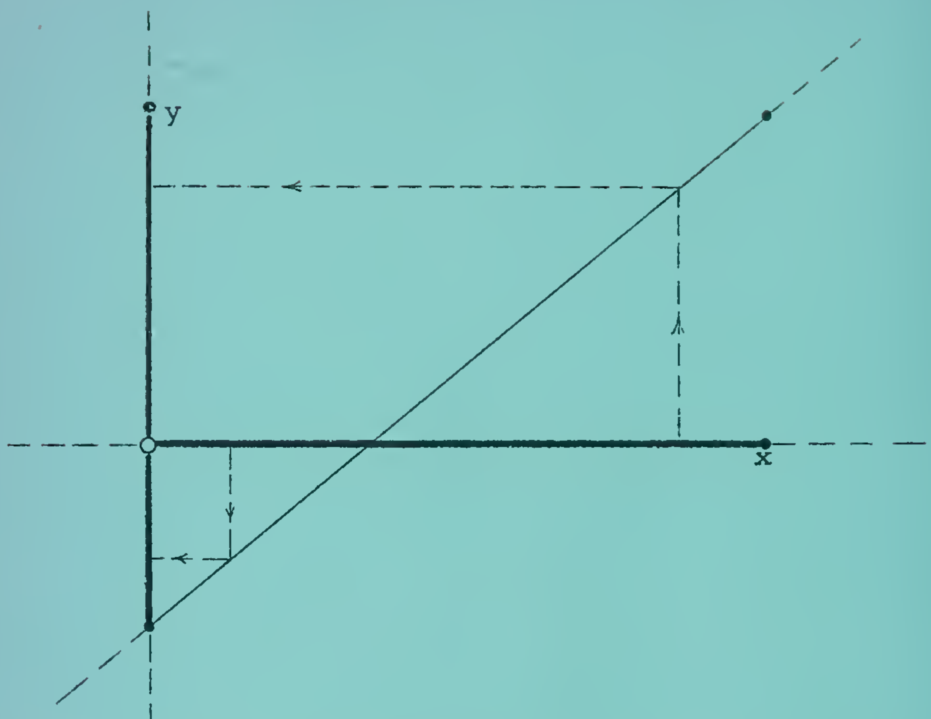
What is the domain of each of the pronumerals 'P' and 'W' shown in the chart?

7. The U. S. Post Office has a "book rate" of 8 cents for the first pound and 4 cents for each additional pound or part of a pound. Make a chart for determining the postal charge for all book shipments which do not exceed 15 pounds in weight.

(a) How much is the postage charge for a 12-pound package of books?

(b) What is the minimum weight and what is the maximum weight of packages which require exactly 36 cents postage?

(continued on next page)



then push each of the points from the line across to the y-axis, they obtain the new interval for which they are looking.

* * *

Exercise 10 gives an example where the student-method first discussed for Exercise 9 does fail. If the student replaces each 'x' by a '-2', and solves for 'y', he obtains 4. Similarly, if he replaces each 'x' by a '2', and solves the equation for 'y', he also obtains 4. Thus, he would conclude that the domain of 'y' is the set of all numbers between 4 and 4! Of course, the student who used the brief method in Exercise 9 probably would not have difficulty with Exercise 10 because he would realize that his method did not apply. However, if students draw a locus [or even plot just a few points], it will be easy for them to see that the domain of 'y' is the set of all numbers between 0 and 4.

In (c) of Exercise 7 the word 'exactly' is important. 22 cents postage would carry a package which weighed 4 pounds, and 2 cents would be wasted; but since we use the word 'exactly', we intend the student to answer that there is no weight which requires this postage. The same analysis holds for part (d). For parts (c) and (d) you might want to ask the related question: What are the minimum weight and the maximum weight of packages which could actually be shipped with this amount of postage on them?

* * *

In Exercise 8, the class will have to decide what kind of numbers can be used in giving the age of a child. Certainly it is possible for a child to be $1\frac{7}{12}$ years old, but this is not a customary way of giving ages. Perhaps the class will decide that the appropriate kind of numbers to use are whole numbers larger than 0. If this is the decision, the domain of each of the pronumerals will be the set $\{1, 2, 3, 4, 5, 6, \text{and } 7\}$. In this case, the locus is a discrete set of points. You might also want to do the problem over again where the domain of each of the pronumerals is the set of all numbers between 0 and 8.

* * *

In Exercise 9, some students will probably be able to answer the question without making a graph. They will merely replace 'x' by a '0', and solve for 'y'. Then they will replace 'x' by a '10', and solve for 'y' again. Then they will say that the corresponding domain of 'y' is the set of all numbers between the two numbers they have just found. This is correct, but there remains the question of why this method gives a correct result and whether it would give a correct result when the locus of the equation is not a straight line. In order to understand this problem, students should draw the locus of the equation, and should indicate on the x-axis the set of all points with first coordinate between 0 and 10. Then, students should develop an intuitive feeling that if they "push" each of the points in this interval on the x-axis down (or up) to the line, and

(continued on T. C. 48B)

(c) What is the minimum weight and what is the maximum weight of packages which require exactly 22 cents postage?

(d) What is the minimum weight and what is the maximum weight of packages which require exactly 27 cents postage?

8. The sum of the numbers of years in the ages of two children is 8. Give a formula for finding the age of one of the children when you know the age of the other. Tell the domain of each of your pronumerals. Make a graph for this formula.

9. Consider the equation:

$$2x - 3y = 7.$$

If the domain of 'x' is given by:

$$0 < x < 10$$

what is the corresponding domain of 'y'?

10. Consider the equation:

$$y = xx.$$

If the domain of 'x' is given by:

$$-2 \leq x \leq 2$$

what is the corresponding domain of 'y'?

(continued on next page)

6. Which of the following statements is true? $[6^2 = 6 \times 6.]$

a) $\sqrt{6^2 + 10^2} = 6 + 10$ b) $\sqrt{(6 + 10)^2} = 6 + 10$

c) $\sqrt{6^2 - 10^2} = (6 - 10)(6 + 10)$ d) $\sqrt{6^2 \times 10^2} = (6 \times 10)^2$

e) $\sqrt{6 - 10} = (6 - 10)^2$

7. The distance from the graph of (-100007, + 382) to the graph of (-100007, -369) is the same as the distance from the graph of

a) (+382, -100007) to the graph of (+369, + 100007).
b) (-382, +100007) to the graph of (-369, -100007).
c) (+100007, +369) to the graph of (+100007, -382).
d) (-100007, +382) to the graph of (+100007, +369).
e) (+100007, -382) to the graph of (-100007, -369).

8. The average of five equal scores is 18 more than 3 times one of the scores. What is the sum of the five scores?

9. The locus of $\{(a, b): a = 9 \text{ and } b = -5\}$ consists of:

a) one line only b) two lines only c) two points only
d) one half-line e) one point only

10. The second satellite launched by the Russians (popularly called 'Muttnik) was reportedly 500 miles farther out in space than Sputnik. If these satellites had travelled in circular orbits, the orbit of Muttnik would have been how much longer than that of the Sputnik?

As you begin the Review Exercises, emphasize the initial instructions. In these exercises the students will learn many new things and they should be warned to expect it.

* * *

In Exercise 2 it is quite acceptable [in fact, it shows insight] if the student gives the two equations '$x = 2$' and '$y = 3$'. If you want the student to "do more work" than this, ask him to find more than one pair of equations whose loci have the desired intersection. But be sure that all students see that this pair of simple equations is a correct answer to the exercise.

* * *

Quiz.

1. The graph of (2, 7) is a point in the graph of which of these sets?

   a) $\{(a, b): 3a + 2b = 25\}$ b) $\{(m, n): 2m + 3n = 25\}$

   c) $\{(r, s): 3r - 2s = 25\}$ d) $\{(x, y): 2x - 3y = 25\}$

   e) $\{(d, e): -2d + 3e = -25\}$

2. The locus in (c, d) of which of the following does not contain the origin?

   a) $5c + 3d = 0$ b) $3cc + 2d = 8cd$

   c) $d = -3c$ d) $1.5c + 2.5d = \frac{32cd}{8}$

   e) $c + 5 = 3d + 4$

3. If the circumference of a circle is πk inches then the area of that circle is ______ square inches.

4. If the lengths of the two smaller sides of a right triangle are 1.3 inches and 2.7 inches respectively, what is the length of the hypotenuse?

5. There are two numbers whose average is 21. One of the numbers is -17 What is the other number?

(continued on T. C. 49C)

Exercise 11 may seem baffling to the student because of its wordiness. Be sure that the student understands how the sample ordered pair (10, 265) is obtained. The 10 is chosen arbitrarily, (or almost arbitrarily). Then the student thinks: "The first component in the ordered pair tells me the number of cows the farmer might own; therefore, I know that in this case he owns 10 cows and therefore, I know that the cows contribute 40 to the total number of legs. Since there are 570 legs in all, I know that the chickens must contribute 530 legs and it takes 265 chickens to contribute 530 legs. Therefore, the second component of my ordered pair is 265." The student should find that the largest possible number of cows is 142 because this many cows contribute 568 legs. Therefore, the smallest possible number of chickens is 1. On the other hand, the smallest possible number of cows is 0, because there could be 285 chickens and they would contribute the necessary 570 legs. Thus, he finds that the domain of the pronumeral for the number of cows is $\{x\colon x \text{ is a whole number and } 0 \leq x \leq 142\}$.

Some students may be tricked and argue analogously that the domain of the pronumeral for the number of chickens is $\{x\colon x \text{ is a whole number and } 1 \leq x \leq 285\}$. But if the student reflects for a minute, he will realize that he has many numbers in this domain which could not be the number of chickens. For example, try 2 chickens. They contribute 4 legs and the remaining number of cows' legs is 566. But, there is no number of cows which can contribute 566 legs because 566 is not exactly divisible by 4. So, the correct domain of the pronumeral for the number of chickens is $\{x\colon x \text{ is an odd number and } 1 \leq x \leq 285\}$.

If your students enjoy this problem, there are endless variations on it. For example, you can consider a room in which there are three-legged stools and four-legged chairs and some tables with six legs. Then you can give the total number of furniture legs in the room and begin on a long list of questions as in Exercise 11. [You would need to consider ordered triples instead of ordered pairs.] Or, you can talk about a printed page on which the only kinds of words are words with 1 letter or 3 letters or 7 letters and give the total number of letters on a page.

You will recognize that the equations which arise from problems such as these are called <u>linear Diophantine equations</u>. Many of the mathematical riddles and puzzles of folklore are linear Diophantine problems. See Chapter 6 of Ore's <u>Number Theory and Its History</u> (New York: McGraw-Hill, 1948).

* * *

(continued on T. C. 49B)

11. A farmer keeps cows and chickens. The total number of legs of these animals is 570. Consider all the ordered pairs in which the first number is a number of cows the farmer might own and the second number is the corresponding number of chickens he would own. For example, one ordered pair is (10, 265). If you listed all such possible ordered pairs, what would be the smallest first number? The smallest second number? The largest first number? The largest second number? If the values of 'x' are numbers of chickens and the values of 'y' are numbers of cows, what are the domains of these pronumerals? Give a formula for finding the number of cows when you know the number of chickens.

REVIEW EXERCISES

In this set of exercises you will find problems which help you review what you have learned in this Unit. Also, you will find exercises reviewing your general knowledge of mathematics. In addition, some exercises will teach you something you did not previously know. For each exercise you should ask yourself, "Should I have learned this in the Unit, did I already know it, or am I learning something new?"

A. Follow the directions in each exercise.

1. In Unit 3 you learned that the locus of '$x = 2$' on the number line is a single point. What is the locus (or graph) of '$x = 2$' on the coordinate plane? What is the locus of '$x > 2$' on the number line? What is the locus of '$x > 2$' on the coordinate plane?

2. Write two equations whose graphs in the coordinate plane intersect in the graph of (2, 3).

(continued on next page)

The student needs to realize that his job, in general, is to find two equations whose loci in the number plane intersect in points such that each point has at least one non-whole number component.

* * *

For Exercise 6, the student should recognize that if two equations have loci which coincide, they must each be satisfied by the same pairs of numbers because it was these pairs of numbers which determined the loci. But, two equivalent equations in two pronumerals are two equations which are satisfied by exactly the same pairs of numbers. [As before, a pair of equations such as 'y = 3' and '2y = 6' is a satisfactory answer.]

* * *

We think students will do well on Exercises 7 and 8. Expect answers to Exercise 7 like 'x + y = x + y' or even like 'x = x'. Although probably none of your students will think of it, it would be correct to give an equation like '2 = 2'. If it is hard to convince them that for every replacement of 'x' and 'y' in the equation '2 = 2', the resulting equation is a true statement, ask them if they consider the equations '$0 \cdot x + 0 \cdot y + 2 = 2$' and '2 = 2' to be equivalent equations. $\{(x, y): 2 = 2\}$ is the number plane.

* * *

In Exercise 8 most students will probably give an equation like 'x + y = x + y + 1'. Here, again, it would be correct to give the equation, say, '2 = 1'. For every replacement of 'x' and 'y' in the equation '2 = 1', the resulting equation is a false statement. $\{(x, y): 2 = 1\} = \emptyset$.

* * *

Let students tackle Exercise 9 without any preliminary coaching except to urge them to read the exercise carefully. We want the student to discover for himself that in (a), for example, the efficient way to proceed is to replace the 'x' by a '3' and also to replace the 'y' by a '5', and then to solve the resulting equation for 'k'. You might challenge your better students to make up an example for Exercise 9 in which two different values of 'k' work. In an equation such as 'x + ky = 13', where you contemplate replacing the pronumerals by numerals in two stages--that is, where first you will replace 'k' by a numeral and then you will, say, draw the locus of the resulting equation--the pronumeral 'k' is called a parameter. Thus, a parameter is a pronumeral which is to be replaced before the other pronumerals in an expression.

In discussing Exercise 3 the class should discover the general characteristics of an equation whose locus contains the origin. An interesting procedure to follow in class is this: Assign Exercise 3 as homework without any particular emphasis or warning. Then, when students come to class, write 8 or 10 equations (non-linear as well as linear) on the board such that approximately half of them have loci which contain the origin, and the other half have loci which do not contain the origin. Ask students to separate these equations into the two categories. Let us know in your daily reports approximately what part of your class was able to do this immediately. Of course, you will want to discuss this question so that the entire class will know the general characteristics of equations whose loci contain the origin.

* * *

A pair of equations such as '$x = 3$' and '$x = 4$' is satisfactory as an answer to Exercise 4. Let students struggle to find pairs of equations whose loci do not intersect. If you want to develop a feeling that "straight lines with equal slopes are parallel", suggest that each of these pairs of equations:

$y = x$	$y = 2x$	$y = 5x + 7$
$y = x + 1$	$y = 2x + 3$	$y = 5x - 3$

gives a pair of loci which are parallel straight lines. Let the student "check" by drawing loci. Then, write a single equation, say, '$y = 3x + 5$' and ask students to try to guess equations whose loci are straight lines which are parallel to the locus of '$y = 3x + 5$'. All you should want to accomplish here is the development of the feeling that equations of the form:

$$y = mx + b$$

have parallel loci if 'm' has the same value for each of the equations. It is unnecessary to formalize the discussion to the extent of defining 'slope'.

* * *

An insightful answer for Exercise 5 is:

$$'x = \frac{1}{2}' \text{ and } 'y = \frac{1}{2}'.$$

(continued on T. C. 50B)

3. Write three equations whose graphs in the coordinate plane intersect in the origin.

4. Write two equations whose graphs in the coordinate plane do not intersect.

5. Consider (1) a plane lattice whose points have all possible whole number coordinates, and (2) a coordinate plane. Write two equations whose graphs on the lattice do not intersect but whose graphs on the coordinate plane do intersect.

6. Write two different equations whose graphs coincide.

7. Write an equation whose graph is the set of all points in the coordinate plane.

8. Write an equation whose graph in the coordinate plane is the empty set.

9. For each of the following replace 'k' by a numeral such that the graph of the resulting equation will include the graph of the ordered pair of numbers which is also given.

   (a) $x + ky = 13$; $(3, 5)$ (b) $5x - 2y = k$; $(-2, -3)$

   (c) $2ky - 7x = 15$; $(4, 7)$ (d) $3xx - 5ky = 12k$; $(2, 5)$

10. Read each of the following statements and tell whether it is true or false.

   (a) $2 > 5$ (b) $5 > 2$

   (c) $2 \geq 2$ (d) $2 \leq |-2|$

   (e) $|-4| \leq 0$ (f) $|6 - 6| = 0$

(continued on next page)

(g) $|6| + |-6| = 0$ (h) $|6| - |-6| = 0$

(i) $\frac{-15}{-3} = 5$ (j) $\frac{-15}{3} = -5$

(k) For every x and y,

$$|x - y| \geq |x| - |y|$$

(l) For every x and y,

$$|x + y| \leq |x| - |y|$$

B. Give equivalent expressions which are simpler.

1. $\frac{-12b}{-3}$
2. $b + c + b + c + c$
3. $(-4)(-8a)$
4. $(-2a)(30a)$
5. $(4a)(-2)(-5x)$
6. $72y \div (-3)$
7. $\frac{4}{5}\square + \frac{1}{10}\square$
8. $\frac{x}{4} \times \frac{2y}{3x}$, $[x \neq 0]$
9. $3(x - 2y) + 5(2x - 3y) + 4(5x + 7y)$
10. $2(a - 3b) - 7(2a - 4b) - 6(3a + b)$
11. $2x(3x - 5y) - 7y(2y - 3x)$
12. $5m(2m - 6q) - 5q(2m + 6q)$

C. Consider a plane lattice whose points are the graphs of every ordered pair of whole numbers (x, y) such that $-10 < x < 10$ and $-10 < y < 10$. Now consider the points of this lattice which are the points in each of the following sets

Set I: The graph of every (x, y) such that

$$x = y.$$

Set II: The graph of every (x, y) such that

$$x = 3.$$

(continued on next page)

In each of Exercises 9 and 10 of Part C, the students are working with 3 sets. We have defined neither the union nor the intersection of 3 sets. However, students should be able to suggest on their own that the union of 3 sets is the set of all points which are contained in any of the 3 sets, and that the intersection of the sets is the set of all points common to all 3 sets.

* * *

Notice that we are saying that an exponent is a number, and that we call the mark on paper which names that number an <u>exponent symbol</u>. We don't expect you to make too much of a fuss with the students about the semantic aspects of exponents and powers. However, you should know the way we view this matter. A power is a number, an exponent is a number, and a base is a number. We say that the number 8^3 is the third power of 8 or that the power 8^3 has the exponent 3 for the base 8. The numeral '8^3' contains the exponent symbol '3' and the base symbol '8'. Since $8^3 = 2^9$, it is also correct, although a little far-fetched, to say that the power 8^3 has the exponent 9 for the base 2. On the other hand, it is obviously incorrect to say that the numeral '8^3' contains the exponent symbol '9' and the base symbol '2'. [We have not introduced 'base' in the students' material because it is not needed in this introductory treatment. THIRD COURSE has a full unit on exponents and logarithms which treats these questions in much detail. In that unit we use pronumerals to stand in place of exponent symbols.]

* * *

Set III: The graph of every (x, y) such that

$$y = 2.$$

Set IV: The graph of every (x, y) such that

$$x + y = 6.$$

Set V: The graph of every (x, y) such that

$$-5 \leq x \leq -1 \text{ and } 1 \leq y \leq 7.$$

Complete the following table by giving the numbers of points in the intersections and unions.

	Sets	Number of points in Intersection	Union		Sets	Number of points in Intersection	Union
1.	I, II			2.	III, IV		
3.	III, V			4.	II, V		
5.	I, IV			6.	II, III		
7.	I, I			8.	V, V		
9.	I, II, III			10.	I, II, IV		

D. Exponent-symbols and powers.

You recall that in arithmetic the symbol:

$$5 + 5 + 5 + 5$$

could be abbreviated as:

$$4 \times 5$$

and the symbol:

$$3 + 3 + 3 + 3 + 3 + 3$$

as:

$$6 \times 3.$$

It is also customary to abbreviate a symbol such as:

$$5 \times 5 \times 5 \times 5$$

by writing:

$$5^4.$$

7. $13^{-5} \times 13^{-4} \times 13^{-7}$

8. $7^{-1} \times 8^{-5} \times 7^{-3} \times 8^{-3}$

9. $4^{-2} \times 6^{-3} \times 4^{7} \times 6^{5}$

10. $\dfrac{11^8}{11^3}$

11. $\dfrac{15^{14}}{15^{12}}$

12. $\dfrac{7^2}{7^8}$

13. $\dfrac{6^4 \times 3^5}{6^7 \times 3^2}$

14. $\dfrac{9^{-5}}{9^{-6}}$

15. $\dfrac{19^{-7}}{19^{-12}}$

16. $\dfrac{8^{-5} \times 3^4}{8^{-4} \times 3^{-5}}$

17. $7^6 \times 7^{-6}$

18. $9^{-3} \times 9^3$

19. $28^{-71} \times 28^{71}$

20. $107^{-2} \times 107^2$

21. $3 \times 10^6 \times 2 \times 10^7$

22. $3.1 \times 10^{-9} \times 2.5 \times 10^6$

23. $\dfrac{8.1 \times 10^7 \times 3.4 \times 10^{-6}}{2.5 \times 10^{-3} \times 6.1 \times 10^8}$

* * *

Ask students to note that

$7^3 = 7 \times 7 \times 7$ | $9^3 = 9 \times 9 \times 9$

$7^2 = 7 \times 7$ | $9^2 = 9 \times 9$

$7^{-1} = \frac{1}{7}$ and $9^{-1} = \frac{1}{9}$

$7^{-2} = \frac{1}{7} \times \frac{1}{7}$ | $9^{-2} = \frac{1}{9} \times \frac{1}{9}$.

In view of these facts, can they suggest definitions of '7^1', '7^0', '9^1', and '9^0'? These expressions are not abbreviations, but are defined in such a way that the same principle which tells a student that $7^{85} \times 7^{41} = 7^{126}$ and that $3^{-5} \times 3^{-6} = 3^{-11}$ and that $9^6 \times 9^{-4} = 9^2$ can also be used in telling students that $7^5 \times 7^1 = 7^6$, $7^5 \times 7^0 = 7^5$, and $9^6 \times 9^{-6} = 9^0$.

In checking the eight statements for correctness, the student <u>must</u> use the abbreviation idea. Do not give him any of the customary "laws of exponents". If he induces rules that work, he should not be discouraged from using them. However, he should be required to demonstrate, for example, that $3^4 \times 3^5$ equals 3^9 because of the meaning of '3^4', '3^5', '3^9', and the associative principle for multiplication, and <u>not</u> because $4 + 5 = 9$.

* * *

As preparation for the exercises of Part D, Miss McCoy suggests that in addition to the eight statements given, the following two should be added:

(9) For every x and y, $xy^2 = x \times y \times y$

(10) For every x and y, $(xy)^2 = x \times y \times x \times y$

* * *

Give the students a brief introduction to negative (whole) number exponents. Just as '$7 \times 7 \times 7$' is abbreviated to '7^3', so a term such as '$\frac{1}{7} \times \frac{1}{7} \times \frac{1}{7}$' is abbreviated to '$7^{-3}$'. Students should then simplify the following expressions.

1. $\frac{1}{5} \times \frac{1}{5}$

2. $\frac{1}{8} \times \frac{1}{8} \times \frac{1}{7} \times \frac{1}{7} \times \frac{1}{7}$

3. $5 \times 5 \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$

4. $6 \times 6 \times 6 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$

5. $9^{-2} \times 9^{-3}$

6. $8^{-4} \times 8^{-2} \times 8^{-3}$

(continued on T. C. 53B)

The symbol '4' in '5^4' is called an <u>exponent symbol</u>. Notice how exponent symbols are used in the following statements.

You can write '3^2' as an abbreviation for '3×3'.

You can write '2^3' as an abbreviation for '$2 \times 2 \times 2$'.

You can write '4^4' as an abbreviation for '$4 \times 4 \times 4 \times 4$'.

You can write 'x^2' as an abbreviation for '$x \times x$'.

You can write 'y^3' as an abbreviation for '$y \times y \times y$'.

The number 5^2 is sometimes called the <u>second power of 5.</u> The number 81 is sometimes called the fourth power of 3 because $3^4 = 3 \times 3 \times 3 \times 3 = 81$. The second power of a number is also called the <u>square of the number</u>, and the third power of a number is also called the <u>cube of the number</u>. A symbol such as '3^7' is pronounced "3 to the seventh" or as "3 to the seventh power" or as "the seventh power of 3".

Study the following statements until you are convinced that they are correct.

(1) $3^2 \times 2^3 = 3 \times 3 \times 2 \times 2 \times 2$

(2) $(5 \times 4)^2 = (5 \times 4) \times (5 \times 4) = (5 \times 5) \times (4 \times 4) = 5^2 \times 4^2$

(3) $8^3 \times 8^2 = 8 \times 8 \times 8 \times 8 \times 8 = 8^5$

(4) $5^2 \times 6^3 \times 5^4 \times 6^8 = 5^2 \times 5^4 \times 6^3 \times 6^8 = 5^6 \times 6^{11}$

(5) $\dfrac{4^7}{4^3} = \dfrac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4} = 4 \times 4 \times 4 \times 4 = 4^4$

(6) $\dfrac{5^9}{5^4} = \dfrac{5^4 \times 5^5}{5^4} = 5^5$

(7) For every x and y,

$$(x^3 \times y^2)^2 = x^3 \times y^2 \times x^3 \times y^2 = x^6 \times y^4 = x^6y^4$$

(8) For every z, $z^3 \cdot z^2 \cdot z = z^5 \cdot z = z^6$.

Sort the expressions on the next page into sets of equivalent expressions such that all the expressions (and only those) equivalent to a given one are in the same set with it.

You will recognize that sorting the expressions on this page into sets of equivalent expressions is a standard kind of exercise which we have used frequently. We think that such an exercise develops in a less boring than usual fashion a considerable facility in manipulation and in rapid recognition of equivalent expressions.

Some students will undoubtedly have in their subsets a few "stray sheep", expressions which ought to be included in another set but which they failed to recognize as equivalent to the expressions in that other set. For example, suppose a student (for some reason) thinks that 'x^3y^4' and '$(x^2y)^2y$' belong to the same subset. To convince him that he is wrong, ask him to select values for 'x' and 'y' (other than -1, 0, and 1) and replace the pronumerals, or remind him that

'x^3y^4' is an abbreviation for 'xxx yyyy'

and that

'$(x^2y)^2y$' is an abbreviation for 'xxyxxyy'
which is equivalent to 'xxxx yyy'.

Clearly, it is <u>not</u> the case that for every x and y, xxx yyyy = xxxx yyy.

* * *

Both Mr. Dietz and Mr. Marston reported the need for extra work on exponents in Part D. We suggest that you consult the 1956-57 edition of THIRD COURSE for suggestions of types of exercises which you can give students. Include exponentials with exponent symbols for negative numbers and 0.

* * *

Note that in the list of expressions on page 4-54 we have not included fractions in which the denominator is an exponential with pronumeral base symbol. The reason for not including such expressions in this particular listing is that the zero-restriction on denominators upsets the sorting. If you write additional exercises to get more practice in working with powers, you should include expressions such as:

$$\frac{x^5}{x^4}, \qquad \frac{y^3x^4}{y^2x^7}, \qquad \text{and:} \quad \frac{a^2b^2c^3}{b^3c^2}.$$

Naturally, you will want to mention zero-restrictions on denominators in cases such as these, or suggest the convention mentioned on T. C. 65C of Unit 2.

$y \times y^5$ $\qquad$ x^3y^4 $\qquad$ $(\frac{8}{2})^2$ $\qquad$ square of 6 $\qquad$ $y \times (y^2)^2$ $\qquad$ 2×2^2

$(2 \times 3)^2$ $\qquad$ $x^2 \times x$ $\qquad$ $y^2 \times y^2 \times y^2$ $\qquad$ $xy^2x^2y^2$ $\qquad$ $x \cdot x^2 \cdot x$ $\qquad$ $4 + 2$

$y^5 \times y$ $\qquad$ $2^5 \div 2$ $\qquad$ x^3 $\qquad$ y^6 $\qquad$ $x^2 \times x^2$ $\qquad$ $x^3 \times x$ $\qquad$ 4^2 $\qquad$ y^5

$2 \times 2 \times 3^2$ $\qquad$ $y^2 \times y^3 \times y$ $\qquad$ $(y^2)^2$ $\qquad$ x^3 $\qquad$ $\frac{8^2}{2^2}$ $\qquad$ $y^2 \times y^2 \times y$

third power of 2 $\qquad$ $y^3 \times y^2$ $\qquad$ 4×2 $\qquad$ $\frac{(4 \times 3 \times 4 \times 3)^2}{(3 \times 2)^2}$ $\qquad$ $y^4 \times y$

$2^3(y^3)^2$ $\qquad$ square of 4 $\qquad$ $y^2x^2y^2x$ $\qquad$ cube of x $\qquad$ $(\frac{24}{12})^3$ $\qquad$ $x \times x^2$

cube of 2 $\qquad$ $(x^2y)^2y$ $\qquad$ $(x^2)^2y^3$ $\qquad$ 6×6 $\qquad$ fifth power of y $\qquad$ $2^2 \times 3^2$

x to the fourth $\qquad$ $(xy)^3y$ $\qquad$ second power of (2×3) $\qquad$ $\frac{4^2}{2}$ $\qquad$ $(y^2)^3$

$\frac{2^8}{2^4}$ $\qquad$ $x^2y^2x^2y$ $\qquad$ fourth power of 2 $\qquad$ $x^4y^3 \times y^3$ $\qquad$ $3 \times 2^2 \times 3$

$2 \times 2 \times 2 \times 2$ $\qquad$ $2(2)^2$ $\qquad$ x^4y^3 $\qquad$ $x \cdot y^2 \cdot x \cdot y^2 \cdot x$ $\qquad$ $(xy)^3y$ $\qquad$ $y^4 \times y^2$

6^2 $\qquad$ 9×4 $\qquad$ second power of the square of x $\qquad$ 4 $\qquad$ $(xy)^3x$

$x \cdot x^2 \cdot y^4$ $\qquad$ $(x^2)^2$ $\qquad$ $2 \times 2 \times 2$ $\qquad$ $2^2 \times 2^2$ $\qquad$ $x \cdot x \cdot x$ $\qquad$ 36 $\qquad$ $\frac{8^2}{2^3}$

$x(xy^2)^2$ $\qquad$ 8 $\qquad$ 16 $\qquad$ $\frac{12^2}{2^2}$ $\qquad$ $2 \times (2 \times 3^2)$

E. Consider the following "completion" problem: For every x, if the dimensions of a rectangle are x + 3 inches and x + 7 inches, then the area of the rectangle is __________ square inches. If you write '(x + 3)(x + 7)' in the blank space, the statement is correct. However, '(x + 3)(x + 7)' is often not the most useful equivalent expression which could be written in the blank space.

(continued on next page)

Review, if necessary, the statements of the principles of arithmetic. Students should understand and believe that examples of algebraic manipulation can be explained as applications of these fundamental principles. They should be able to cite these principles as justifications for the steps in the expansion of '$(x + 3)(x + 7)$' and for the steps in the six samples on page 4-56. We omit steps in these illustrations to show students that we expect them to omit steps in the exercises which follow.

Let us see how we can <u>expand</u> an expression like '$(x + 3)(x + 7)$' to get an equivalent expression.

First, consider the expression:

$$(5 + 3)(5 + 7).$$

[You know that this symbol is another name for the number 96.]
First, apply the distributive principle:

$$(5 + 3)5 + (5 + 3)7$$

then the commutative principle for multiplication:

$$5(5 + 3) + 7(5 + 3)$$

then the distributive principle:

$$5 \times 5 + 5 \times 3 + 7 \times 5 + 7 \times 3.$$

Finally, we get:

$$25 + 15 + 35 + 21$$

which is another name for 96.

Now, return to the expression:

$$(x + 3)(x + 7).$$

We know that for every x,

$$\begin{aligned} & (x + 3)(x + 7) \\ = \ & (x + 3)x + (x + 3)7 \\ = \ & x(x + 3) + 7(x + 3) \\ = \ & x^2 + 3x + 7x + 21 \\ = \ & x^2 + 10x + 21 \ . \end{aligned}$$

The expression '$x^2 + 10x + 21$' is equivalent to the expression '$(x + 3)(x + 7)$'. [Convince yourself of their equivalence by replacing 'x' by a numeral several times to see if you get in each case two expressions for the same number.]

(continued on next page)

Study the following examples of expansion.

<u>Sample 1</u>.

$(y + 4)(y + 8)$

$(y + 4)y + (y + 4)8$

$y(y + 4) + 8(y + 4)$

$y^2 + 4y + 8y + 32$

$y^2 + 12y + 32$

<u>Sample 2</u>.

$(3x + 2)(5x + 4)$

$(3x + 2)5x + (3x + 2)4$

$5x(3x + 2) + 4(3x + 2)$

$15x^2 + 10x + 12x + 8$

$15x^2 + 22x + 8$

<u>Sample 3</u>.

$(x - 2)(x + 3)$

$(x - 2)x + (x - 2)3$

$x(x - 2) + 3(x - 2)$

$x^2 - 2x + 3x - 6$

$x^2 + x - 6$

<u>Sample 4</u>.

$(y - 4)(y - 8)$

$y(y - 4) - 8(y - 4)$

$y^2 - 4y - 8y + 32$

$y^2 - 12y + 32$

(Notice that we omitted the first step.)

<u>Sample 5</u>.

$(x - 3)^2$

$(x - 3)(x - 3)$

$x(x - 3) - 3(x - 3)$

$x^2 - 3x - 3x + 9$

$x^2 - 6x + 9$

(continued on next page)

* * *

Note how many of your students fail to simplify the binomials first before multiplying in Exercises 31-34. Point out that these exercises can be handled in more than one way and that usually there is an easiest way. For example, they should agree that in Exercise 34 it is easier to find that

$$(60 - 5)(60 + 5) = 3575$$

by expanding:

$$(60 - 5)(60 + 5) = 3600 - 25$$

than by simplifying first and then doing a long multiplication

$$(60 - 5)(60 + 5) = 55 \times 65.$$

This type of computational short-cut is treated on page 4-58.

As far as possible, let the student discover the short-cut methods through his own efforts. If he needs more simple exercises like Exércises 1-14 in order to discover the short-cuts, you can supply them readily. But, he must learn a short-cut. You might try this device to help the students:

$$(x + 2)(x + 5) \longrightarrow x^2 + \square x + 10$$

$$(x + 3)(x + 4) \longrightarrow x^2 + 7x + \square$$

$$(x - 4)(x - 3) \longrightarrow x^2 + \square x + \bigcirc$$

$$(x + 4)(x + 4) \longrightarrow x^2 + \square x + 16$$

$$(x - 4)(x - 4) \longrightarrow x^2 - 8x + \square$$

* * *

You will note that in each of the first sixteen exercises the expanded form will have a first term whose "numerical" coefficient is '1'. It is a good idea to discuss the following exercises along with the first sixteen:

1. $(3r - 7)(r + 2)$
2. $(n + 10)(2n + 3)$
3. $(a + 8)(4a - 3)$
4. $(5s - 6)(s - 5)$
5. $(2b - 3)(3b - 2)$
6. $(8y + 1)(3y - 2)$

Then you may want to try this device:

$$(\triangle a + 3)(a + 5) \longrightarrow \triangle a^2 + (5\triangle + 3)a + 15$$

$$(b + 6)(\triangle b - 2) \longrightarrow \triangle b^2 + (-2 + 6\triangle)b + 12$$

$$(\triangle r - 7)(\bigcirc r + 2) \longrightarrow \triangle\bigcirc r^2 + (-7\bigcirc + 2\triangle)r - 14$$

$$(\triangle m + 3)(\bigcirc m - 8) \longrightarrow \triangle\bigcirc m^2 + (3\bigcirc - 8\triangle)m - 24$$

(continued on T. C. 57B)

Sample 6. $(3x - 2)(3x + 2)$

$3x(3x - 2) + 2(3x - 2)$

$9x^2 - 6x + 6x - 4$

$9x^2 - 4$

Expand the following expressions. At first you should follow the procedure illustrated in the samples. After a while you ought to develop speed and short-cuts so that you can write the expanded expression immediately, doing all of the intermediate steps mentally.

1. $(x + 2)(x + 5)$
2. $(y + 4)(y + 3)$
3. $(z + 1)(z + 7)$
4. $(u + 6)(u + 8)$
5. $(x - 3)(x + 4)$
6. $(y - 6)(y + 7)$
7. $(x - 2)(x - 8)$
8. $(u - 3)(u - 5)$
9. $(x - 12)(x + 4)$
10. $(y - 17)(y + 2)$
11. $(x + 4)(x - 4)$
12. $(y - 3)(y + 3)$
13. $(x + 5)^2$
14. $(y + 7)^2$
15. $(a - 6)^2$
16. $(b - 3)^2$
17. $(3a - 2)(4a + 7)$
18. $(2y - 3)(5y + 6)$
19. $(7s - 4)(3s - 2)$
20. $(5r - 7)(4r + 2)$
21. $(x - y)(x + y)$
22. $(a - b)(a + b)$
23. $(3x - y)(2x + y)$
24. $(5x - 7y)(3x + 2y)$
25. $(4r - 3s)(7r + 2s)$
26. $(8t - s)(2t + s)$
27. $(3 - 7x)^2$
28. $(5 - 8y)^2$
29. $(3a + 2b)^2$
30. $(6c + 5d)^2$
31. $(10 - 5)(10 + 5)$
32. $(10 - 2)(10 + 2)$
33. $(40 - 3)(40 + 3)$
34. $(60 - 5)(60 + 5)$

The technique of expanding expressions like '$(x - y)(x + y)$' can be used to good advantage in multiplying certain pairs of numbers. We know that '$(x - y)(x + y)$' and '$x^2 - y^2$' are equivalent expressions. Now, consider the problem of multiplying 65 by 75.

If your students are eager for more computational short-cuts, show them the one based on the following generalization:

$$\text{For every } x,\ (x + \tfrac{1}{2})^2 = x(x + 1) + \tfrac{1}{4}.$$

Some applications of this principle are:

$$(7\tfrac{1}{2})^2 = 7 \times 8 + \tfrac{1}{4} = 56\tfrac{1}{4},$$

$$(9.5)^2 = 9 \times 10 + .25 = 90.25,$$

$$(35)^2 = (3 \times 4) \times 10^2 + 25 = 1225.$$

For other computational short-cuts see pages 78-81 of Larsen's <u>Arithmetic For Colleges</u> (New York: Macmillan, 1950).

* * *

Here are supplementary exercises for Part E.

(1) 108×92	(2) $12\frac{1}{3} \times 11\frac{2}{3}$	(3) 34×26
(4) 18.5×21.5	(5) 81×99	(6) 195×205
(7) 78×82	(8) $7\frac{5}{6} \times 8\frac{1}{6}$	(9) 16×24
(10) 51.5×48.5	(11) $10\frac{1}{5} \times 9\frac{4}{5}$	(12) 6.25×5.75
(13) 17×23	(14) 20.125×19.875	(15) 7.8×8.2

* * *

Note that in Part F we use the word 'factor' to refer both to numbers and expressions. This is an unfortunate duality in usage, but we must yield to convention in this case. If we could find a commonly used mathematical word which is an antonym for 'expand', and which refers exclusively to symbols, we might use that word instead of 'factor'.

We notice that

$$65 = 70 - 5 \quad \text{and} \quad 75 = 70 + 5.$$

Therefore,

$$\begin{aligned} 65 \times 75 = (70 - 5)(70 + 5) &= 70^2 - 5^2 \\ &= 4900 - 25 \\ &= 4875. \end{aligned}$$

Simplify mentally by the method explained above.

35. 45×55
36. 85×95
37. 67×73
38. 38×42
39. 97×103
40. 26×34
41. $5\frac{1}{2} \times 6\frac{1}{2}$
42. $8\frac{1}{2} \times 9\frac{1}{2}$
43. $6\frac{3}{4} \times 7\frac{1}{4}$
44. $7\frac{7}{8} \times 8\frac{1}{8}$

F. At times in your study of mathematics you will need to look at an expression like '$x^2 - 5x + 6$' and be able to write the equivalent expression '$(x - 3)(x - 2)$'. That is, you will need to reverse the process you learned in Part E. The process of going from an expression like '$x^2 - 5x + 6$' to the equivalent expression '$(x - 3)(x - 2)$' is called factoring. When you go from a simple name such as '12' to an equivalent name such as '4×3' or '2×6' or '12×1', you are factoring the number 12. The pairs, 4, 3 and 2, 6 and 12, 1 are pairs of factors of 12. Similarly, the pair of expressions '$x - 3$' and '$x - 2$' is called a pair of factors of '$x^2 - 5x + 6$'.

To factor an expression it is often necessary to use a trial-and-error procedure. For example, let us find a pair of factors of '$x^2 + 10x + 24$'. We suspect that the given expression was

Insofar as possible, let students discover their own methods for finding factors in these exercises. Even though it may take your students a little longer to do the exercises, it will be worth the time to let them hunt out methods for themselves. However, you may need to use some of the standard teaching devices to get proficiency in factoring from all of your class. Proficiency in factoring trinomials is closely related to proficiency in applying short-cut methods to the expansion of expressions of the form: $(x + a)(x + b)$.

* * *

Try this device for determining factors of trinomial expressions. It illustrates forcefully the role of the distributive principle.

a) $x^2 + 5x + 6 = x^2 + \square\, x + \bigcirc\, x + 6$

$= (x + \square\,)x + (x + \square\,)\bigcirc$

$= (x + \square\,)(x + \bigcirc\,)$

b) $x^2 - 5x - 24 = x^2 + \square\, x + \bigcirc\, x - 24$

$= (x + \square\,)x + (x + \square\,)\bigcirc$

$= (x + \square\,)(x + \bigcirc\,)$

c) $x^2 + x - 12 = x^2 + \square\, x + \bigcirc\, x - 12$

$= (x + \square\,)x + (x + \square\,)\bigcirc$

$= (x + \square\,)(x + \bigcirc\,)$

d) $a^2 - 11a - 28 = a^2 + \square\, a + \bigcirc\, a - 28$

$= (a + \square\,)a + (a + \square\,)\bigcirc$

$= (a + \square\,)(a + \bigcirc\,)$

obtained by simplifying an expression of the form:

$$(x \quad)(x \quad).$$

We note that the third term in the expression:

$$x^2 + 10x + 24$$

is '24'. What are pairs of factors of 24? 2 and 12, 3 and 8. 4 and 6. We try each pair.

'$(x + 2)(x + 12)$' gives '$x^2 + 14x + 24$'

'$(x + 3)(x + 8)$' gives '$x^2 + 11x + 24$'

'$(x + 4)(x + 6)$' gives '$x^2 + 10x + 24$'.

Therefore, we see that a pair of factors of

'$x^2 + 10x + 24$' are '$x + 4$' and '$x + 6$'.

We say that a factored form of

'$x^2 + 10x + 24$' is '$(x + 4)(x + 6)$'.

Factor each of the following expressions.

1. $x^2 + 10x + 21$
2. $y^2 + 8y + 12$
3. $a^2 + 8a + 15$
4. $k^2 + 9k + 14$
5. $c^2 + 6c + 9$
6. $r^2 + 8r + 16$
7. $z^2 + 11z + 24$
8. $m^2 + 5m + 4$
9. $t^2 - 5t + 6$
10. $s^2 - 7s + 12$
11. $z^2 - 12z + 32$
12. $x^2 - 12x + 27$
13. $b^2 - 3b - 10$
14. $f^2 + 2f - 24$
15. $x^2 + 5x - 24$
16. $x^2 + x - 12$
17. $x^2 - x - 6$
18. $x^2 + 3x - 4$
19. $k^2 - k - 30$
20. $k^2 + k - 30$

(continued on next page)

Whenever students perform manipulations on an expression for the sole purpose of manipulation, they frequently face the question of determining the form of the final expression in a chain of equivalent expressions. This is particularly the case in factoring. For example, in Exercise 40 the student ought to be <u>uncertain</u> about which one of the following expressions the teacher wants as the final expression:

$$4(5x^2 + xy), \quad 4x(5x + y), \quad x(20x + 4y), \quad \text{or even:} \quad 20x(x + \tfrac{1}{5}y).$$

It is a rather difficult task to specify exactly what is required, and it is very easy to point to loopholes in most such specifications. The ultimate goal of factoring [and manipulation, in general] is to obtain an expression whose form is useful for some given purpose. [Examples: (1) factoring an expression in order to solve a quadratic equation; (2) factoring or expanding to provide easy computational devices.] So, instead of holding up some arbitrary and complicated standard, encourage the students to give as many factored forms as they can for Exercises 34-49, and be sure they see the equivalence of the various factored forms for a given expression.

* * *

Some students do not see the relationship between Exercises 34-49 and the distributive principle. In order to show this connection, we suggest that you precede these exercises by a set of exercises similar to those in Part A on pages 2-79 and 2-80. In fact, stress that all of the factoring problems in Part F make use of the distributive principle.

* * *

In Part G we give an opportunity for the student to apply his skill in factoring in order to make computations easier. Exercises 3 and 4 are instances of this. Expand the list of exercises if you care to.

21. $j^2 + 3j - 10$
22. $r^2 + 2r - 35$
23. $n^2 - 10n + 25$
24. $p^2 - 18p + 81$
25. $x^2 - 4$ [Answer: $(x - 2)(x + 2)$]
26. $y^2 - 16$
27. $z^2 - 64$
28. $k^2 - 25$
29. $t^2 - 1$
30. $4x^2 - 9$
31. $100s^2 - 81$
32. $6b^2 + 13b + 6$
33. $8t^2 + 26t + 15$

There are some expressions for which one of the factors is a single term. For example, a factored form of the expression '$x^2 + 3x$' is '$x(x + 3)$'.

Factor these expressions:

34. $5b^2 + 2b$
35. $6ax + 7ay$
36. $2xz - 3xy$
37. $10x^2 + 3x$
38. $2x + 2y$
39. $6x - 3y$
40. $20x^2 + 4xy$
41. $3a^2x^2 - 2ax$
42. $7x^2y - 2xy^2$
43. $2 + 4x$
44. $3x + 6y + 9$
45. $18x^2 + 6y^2 + 3$
46. $3a^2b^2 + 2ab$
47. $x^3 + x^2 + x$
48. $3y^2x^3 + 2yx^2 + 3yx$
49. $a^2z^2 - 2az^3 + 3a^2z$

G. Substitute numerals for the pronumerals as indicated and simplify. Look for shortcuts!

1. $3ab - 2ax - 5by$; 'a' by '2', 'b' by '3', 'x' by '-7', 'y' by '-4'
2. $5(x^2 - y) - 7(y^2 - x)$; 'x' by '-3', 'y' by '4'
3. $\dfrac{x^2 - 2x - 35}{x^2 - 10x + 21}$; 'x' by '$-\frac{2}{3}$', 'y' by '$\frac{-3}{5}$'
4. $2A^4B^2 - 3A^2B^2 + 5A^2B^4$; 'A' by '.01', 'B' by '.002'
5. $\dfrac{x^5 - y^5}{x - y}$; 'x' by '-2', 'y' by '-1'

H. Solve the following equations.

1. $x + 7 = 16$

2. $2x + 5 = 15$

3. $\frac{x}{4} = 4$

4. $\frac{x + 1}{3} = 2$

5. $\frac{x}{3.7} = 0$

6. $\frac{0}{x} = 3.7$

7. $3 - x = x - 3$

8. $x - 3 = -(3 - x)$

9. $3 + x = -x - 3$

10. $4x + 3 = x + 2 + x$

11. $x + \frac{1}{3} = 2\frac{1}{6}$

12. $\frac{1}{4} = \frac{1 - x}{3}$

13. $\frac{x + 15}{3} = 20$

14. $\frac{y}{7.5} = 10$

15. $(x - 1) - (1 - x) = x - 3$

16. $10\% x + 1 = 5$

17. $7(x + 1) = -4(x - 7)$

18. $15\% x = 10\%(x - 1) + x$

19. $\frac{A}{8} = \frac{A}{2} - \frac{A + 1}{4}$

20. $\frac{15y - 24}{7} = \frac{10}{7}$

21. $y + \frac{4}{5}y = 180\% y$

22. $\frac{2x - 5}{7.1} = 0$

23. $3(x - 3) + 7(5 - 2x) = 4(3x - 7) - 2(x - 5)$

24. $8(y + 5) - 6(5 - 3y) = 9(1 - 3y) + 2(8y - 1)$

Sample 1. $(x + 3)(x + 2) = (x + 5)(x - 1)$

Solution.
$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 4x - 5 \\ 5x + 6 &= 4x - 5 \\ x &= -11 \end{aligned}$$

Check:
$$\begin{aligned} (-11 + 3)(-11 + 2) &= (-11 + 5)(-11 - 1) \\ (-8)(-9) &= (-6)(-12) \\ 72 &= 72 \checkmark \end{aligned}$$

(continued on next page)

We are curious to know how many of your students are clever enough to recognize that the equation in Exercise 28 can be solved at once by applying the commutative principle for multiplication. Students who fail to see this and plunge headlong into expanding both members have formed mental sets which frequently block creative thinking. We try to discourage sheerly mechanical approaches to problems and exercises by distributing "foolers" like this one among the exercises. We do want a student to develop considerable mechanical facility, but certainly not at the expense of sacrificing inventiveness or a questioning attitude. We strive to maintain a proper ratio of the number of opportunities for the exercise of inventiveness to the number of opportunities for the exercise of manipulatory skill.

* * *

Students should see that it is easy to solve an equation in which one member is in factored form and in which the other member is '0'. After students have solved several of these equations, give them an equation such as '$(x - 5)(x - 0) = 12$'. Probably, at least a few of your students will "equate each factor to 12" or "equate one factor to 3 and the other to 4" and thereby obtain erroneous results. It is important that they recognize that an essential part of this method of solution is that one member stands for 0.

* * *

In giving the roots for these equations do not permit students to write incorrect statements such as 'The roots are $x = 5$ and $x = -2$'. Simple and correct statements such as 'The roots are 5 and -2' are preferable. [Recall for your students that '$x = 5$' is an <u>equation</u>; one which is easily solved!]

* * *

Note that each of the equations in Exercises 46 and 47 has one and only one root. We do not want a student to be told that, for example, 1 and 1 are roots in Exercise 46. If he has any sense, he will reply that he thought that 1 and 1 and 1 and 1 and 1 are roots. When giving the roots of an equation, it is necessary to mention each root just once. In later courses we shall deal with the idea of the <u>multiplicity</u> of a root, and we will say that the equation '$(x - 1)(x - 1) = 0$' has 1 as a root of multiplicity 2.

25. $(x + 2)(x + 5) = (x + 8)(x + 3)$

26. $(x + 9)(x - 1) = (x + 1)(x - 7)$

27. $(x - 1)(x + 4) = (x - 2)(x + 5)$

28. $(x - 2)(x - 3) = (x - 3)(x - 2)$

29. $(x + 5)(x - 1) - 3(x + 1) = (x + 6)(x - 2) + 8(x - 3)$

30. $(x - 4)(x + 3) + 5(x - 2) = 6(2 + x) + (x + 7)^2$

<u>Sample 2</u>. $(x - 3)(x + 1) = 0$

<u>Solution</u>. You solved equations like this in Unit 3. Recall that you must find replacements for 'x' such that the expression '$(x - 3)(x + 1)$' becomes a symbol for 0. It is easy to see that when 'x' is replaced by '3', we get:

$$\begin{aligned} &(3 - 3)(3 + 1) \\ = &(0)(4) \\ = &0. \end{aligned}$$

So, we have found that one root of the equation is 3. Another root is -1 because

$$\begin{aligned} &(-1 - 3)(-1 + 1) \\ = &(-4)(0) \\ = &0. \end{aligned}$$

31. $(x - 5)(x + 2) = 0$

32. $(x + 5)(x + 9) = 0$

33. $(y + 7)(y + 6) = 0$

34. $(z + 3)(z - 3) = 0$

35. $x(x + 8) = 0$

36. $2y(y - 10) = 0$

37. $(2a + 3)(5a - 6) = 0$

38. $(4x + 7)(2x + 9) = 0$

39. $(3p + 8)(2p - 5) = 0$

40. $(7q - 7)(8q + 8) = 0$

41. $(3 + x)(8 - x) = 0$

42. $(9 - k)(11 + k) = 0$

43. $x(x + 12)(3 - x) = 0$

44. $y(2y + 13)(6 - \frac{1}{2}y) = 0$

45. $(3w - 7)(2w + 7)w(w - \frac{1}{2})(3\frac{1}{2} - 2w)(8 + 5w) = 0$

46. $(x - 1)(x - 1) = 0$

47. $(2x + 6)(x + 3) = 0$

$$x^2 - 5x + \frac{4}{4} + \frac{21}{4} = 0 + \frac{21}{4}$$

$$x^2 - 5x + \frac{25}{4} = \frac{21}{4}$$

$$(x - \frac{5}{2})^2 = \frac{21}{4}$$

The roots are $\frac{5}{2} + \sqrt{\frac{21}{4}}$ and $\frac{5}{2} - \sqrt{\frac{21}{4}}$.

[You can simplify '$\sqrt{\frac{21}{4}}$' to '$\frac{\sqrt{21}}{2}$' and combine into a single term if you like; but this step is relatively unimportant right now.] Give several equations for practice. Use the phrase 'completing the square' so that students will be able to identify the phrase when they see it used in other texts.

Next, consider '$x^2 = 7$'. Here the roots are $\sqrt{7}$ and $-\sqrt{7}$. [Point out that, by convention, '$\sqrt{7}$' is a name for the positive number whose square is 7. There is no simpler name than '$\sqrt{7}$' for this number.] Then, consider '$(x + 4)^2 = 36$'. The two roots in this case are 2 and -10; these roots are discovered by inspection. Treat equations such as '$(y + 5)^2 = 49$', $(z + 3)^2 = 81$', '$(x - 2)^2 = 9$', and '$(x + \frac{3}{2})^2 = 100$'. Now, take '$(x - 5)^2 = 17$'. We seek values of '$x$' such that when 5 is subtracted from each value, the result is a square root of 17. One such value is $5 + \sqrt{17}$; the other is $5 - \sqrt{17}$. Then try '$(x + 4)^2 = 10$'. Return to '$x^2 + 6x + 2 = 0$' and indicate that if this equation could be transformed into one of the form:

$$(x + \ldots)^2 = - - -$$

we could easily find its roots. We can transform it to the desired form by writing a '+7' on both sides:

$$x^2 + 6x + 2 + 7 = 0 + 7$$

$$x^2 + 6x + 9 = 7$$

$$(x + 3)^2 = 7$$

The roots are $-3 + \sqrt{7}$ and $-3 - \sqrt{7}$.

Now, consider:

$$x^2 - 5x + 1 = 0.$$

To transform this equation into one of the form:

$$(x - \triangle)^2 = \square$$

would entail discovering a replacement for the '$\triangle$' such that $x^2 - 2\triangle x = x^2 - 5x$, for every x. Clearly, $\frac{5}{2}$ works. So, we need '$x^2 - 5x + (\frac{5}{2})^2$' or '$x^2 - 5x + \frac{25}{4}$' as the left member of the derived equation. We transform the given equation to achieve this result.

(continued on T. C. 63C)

Notice that we do not give the student a definition of 'quadratic equation'. We merely tell him that equations like those he sees on the page are called quadratic equations. Unless your students press you, no further description is needed at this time. Actually, it is difficult to give a definition which does not contain some kind of loophole. The student needs to get the idea that the pronumeral with the exponent symbol '2' is the key.

You should tell your students that the quadratic equations given here have been carefully contrived so that they yield fairly easily to the factoring method of solution and that there are many quadratic equations where the factoring method is essentially impossible. You may even want to remind them of the quadratic equation '$x^2 + 1 = 0$' which has no roots among the numbers with which they have been working.

* * *

Do not fail to point out that the students have already solved in Unit 3 equations like those in Exercises 56 and 57. In Unit 3 they were given equations like '$x^2 = 4$' and '$s^2 = 81$'. The intuitive method they used in Unit 3 and the factoring method they are learning here are equally valuable in solving equations like these.

* * *

Both Miss Wandke and Mr. Marston reported that students enjoyed solving quadratic equations by the factoring method. We know that students get a great deal of pleasure out of being able to solve a long set of equation exercises in which the computational load is not too heavy but in which there is enough challenge to make the exercises interesting. You may want to write supplementary exercises of this type.

* * *

If time permits, develop a procedure for solving quadratic equations by "completing the square". Start by proposing the equation '$x^2 + 6x + 2 = 0$'. The "factoring method" doesn't work; we need a new technique. Consider immediately the equation '$x^2 = 25$'. The roots of this equation are 5 and -5 because these are the only numbers which when squared yield 25.

(continued on T. C. 63B)

QUADRATIC EQUATIONS

Sample 3. $x^2 + 7x + 10 = 0$

Solution. You can use your knowledge of how to factor an expression to help you solve this equation.

We see that '$x^2 + 7x + 10$' and '$(x + 2)(x + 5)$' are equivalent expressions. Therefore, the equation:

$$(x + 2)(x + 5) = 0$$

has exactly the same roots as the given equations. The roots of the equation '$(x + 2)(x + 5) = 0$' are -2 and -5. The roots of '$x^2 + 7x + 10 = 0$' are -2 and -5, also. Check these roots.

$(-2)^2 + 7(-2) + 10$	$(-5)^2 + 7(-5) + 10$
$= 4 + (-14) + 10$	$= 25 + (-35) + 10$
$= 0$	$= 0$

Note: Equations like the one in Sample 3 and like those which follow are called quadratic equations.

48. $x^2 + 6x - 16 = 0$

49. $x^2 - 2x - 15 = 0$

50. $x^2 + 15x + 56 = 0$

51. $y^2 - 3y - 40 = 0$

52. $k^2 + 12k - 13 = 0$

53. $t^2 + 3t - 4 = 0$

54. $x^2 + 5x = 0$

55. $6x^2 + 7x = 0$

56. $x^2 - 4 = 0$

57. $s^2 - 81 = 0$

58. $6s^2 + 11s + 3 = 0$

59. $28u^2 + 65u + 28 = 0$

(continued on next page)

Sample 4. $x^2 - 9x - 6 = 4$

Solution. You can treat this equation as you did the preceding ones if you first transform it to an equation in which one member is '0' and the other member is in factored form.

$$x^2 - 9x - 6 = 4$$
$$x^2 - 9x - 6 - 4 = 4 - 4$$
$$x^2 - 9x - 10 = 0$$
$$(x + 1)(x - 10) = 0$$
$$x = -1 \text{ or } 10$$

The roots are -1 and 10.

60. $x^2 - 5x - 20 = 4$

61. $y^2 + 7y - 7 = 1$

62. $s^2 - 5 = 4$

63. $12 + x^2 = 37$

64. $x^2 + 5x - 3 = 15 - 2x$

65. $4x - 4 = x^2$

66. $3x^2 - 12x - 58 = 2x^2 - 10x - 10$

I. Solve these equations for the pronumeral indicated.

1. $3a - 2b = 7$; 'a'

2. $5x + 2y = 3$; 'y'

3. $p - \frac{2}{q} + 2p = 8$; 'p'

4. $\frac{1}{x} + y = 9$; 'x'

5. $\frac{1}{r} + \frac{1}{s} = 6$; 'r'

6. $\frac{1}{p} + \frac{1}{q} = \frac{1}{b}$; 'q'

J. Solve these problems.

1. If 5 were added to a certain number, the result would be the same as if the number were divided by 2. What is the number?

(continued on next page)

2. A rectangle and a parallelogram have the same area. The lengths of two sides of the rectangle are 11 feet and 8 feet. The altitude of the parallelogram is $5\frac{1}{2}$ feet. What is the length of the base of the parallelogram?

3. The sum of three consecutive numbers is 51. What are the three numbers?

4. The sum of three consecutive odd whole numbers is 51. What is the smallest of these three whole numbers?

5. Can you find three consecutive odd whole numbers whose sum is 98? (Can the sum of three odd whole numbers ever be an even number?) Write down an equation for this problem similar to the equation you used to solve Exercise 4. Solve it. What is wrong with the solution?

6. Find three numbers whose sum is 15.37.

7. The average of four numbers is 10. The average of the first three of these numbers is $8\frac{1}{3}$. What is the fourth number?

8. The average of ten numbers is 7. What is the sum of the ten numbers?

9. A number is 3 times as large as a second number. If 8 is added to each number, the first number will be 2 times as large as the second number. What is the smaller number?

10. Some theater tickets cost $5.40 including a 20% tax. How much was the tax?